REGENSBURG RESEARCH SEMINAR SS 2020

INTEGRAL HOMOTOPY THEORY (AFTER ALLEN YUAN)

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Introduction

Given a simply connected topological space, its rational and p-adic homotopy types can be understood in terms of the algebras of its cochains. In more detail, rational homotopy theory, due to Sullivan and Quillen, states that associating to a space its cochain algebra with rational coefficients induces a fully faithful embedding from simply connected rational spaces to rational commutative dgas. Later, an analogous statement for the p-adic case was proved by Mandell, with cdgas replaced by E_∞-algebras. However, there was no known way to "glue" the information about rational and p-adic cochains together, in order to reconstruct the integral homotopy type. In his recent paper "Integral models for spaces via the higher Frobenius", Yuan solves this problem by refining the p-adic model, using Nikolaus-Scholze Frobenius map on E_∞-rings as one of the main instruments. In this seminar, we will study the paper [Yua19] by Yuan. We will use the language of modern homotopy theory, the necessary notions from the Nikolaus-Scholze paper will be recalled.

Talks

1. **Historic overview (5.5)**
   Give an overview of rational and p-adic homotopy theory, after Quillen, Sullivan, and Mandell.
2. **Tate construction and Tate diagonal (12.5)**
   Review the Tate construction for finite groups and show that it is lax symmetric monoidal [NS19, I.1, I.3]. Define the Tate diagonal $X \to (X^{\otimes p})^{tC_p}$ [NS19, III.1]. Helpful note on Tate construction: [Hor18].
3. **Equivariant stable homotopy theory (19.5)**
   Review the definition of equivariant stable homotopy theory for finite group(oid)s. Explain the construction of the functor $\mathrm{Sp}^{(-)}\colon \mathrm{Glo}^+ \to \mathrm{Cat}_\infty$ enconding norms and geometric fixed points [Yua19, Sections 2.1-2], see also [BH18, §9].
4. **The E_∞-Frobenius (26.5)**
   Recast the Tate construction and Tate diagonal using equivariant stable homotopy theory [Yua19, Section 2.3]. Define the E_∞-Frobenius and its generalized version [Yua19, Sections 3.1-2].
5. **Borel global agebras (2.6)**
   As motivation, explain the definition of E_∞-ring spectra and of normed (or genuine E_∞-ring) G-spectra using 2-categories of spans [BH18, §9.2, §C.1]. Define global algebras, and show that Borel global algebras are uniquely determined by their underlying E_∞-algebra [Yua19, Sections 3.3-4].
6. **Partial algebraic K-theory: $S_\bullet$-construction (9.6)**
   Define partial K-theory via the $S_\bullet$-construction and show that the group completion of partial K-theory is ordinary K-theory. [Yua19, Section 4.1]

7. **Partial algebraic K-theory: Q-construction (16.6)**
   Define partial K-theory via the Q-construction and shows that it is equivalent to the $S_\bullet$-construction. [Yua19, Section 4.2]
8. **The partial K-theory of $\mathbb{F}_p$ (23.6)**
   Prove Theorem 5.2 computing the partial K-theory of $\mathbb{F}_p$ up to $\mathbb{F}_p$-homology equivalence. [Yua19, Section 5]
9. **The p-complete Frobenius and the action of $B\mathbb{Z}_{\geqslant 0}$ (30.6)**
   Construct the action of $B\mathbb{Z}_{\geqslant 0}$ on the ∞-category of F_p-stable E_∞-rings, encoding Frobenius endomorphisms and their coherences. Give some examples of perfect E_∞-rings. [Yua19, Section 6]
10. **Integral models for the unstable homotopy category (7.7)**
    Prove the main Theorem C: the ∞-category $\mathcal{S}^{\mathrm{fsc}}$ of finite simply connected spaces embeds fully faithfully in the opposite of the ∞-category of Frobenius-fixed perfect E_∞-rings. [Yua19, Section 7]
11. **Grothendieck's schematization problem after Toën (14.7)**
    An overview of [Toë20], which is a different perspective on the topic of the seminar.

References

[BH18] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, 2018, arXiv:1711.03061v4

[Hor18] G. Horel, *Notes on Tate construction*, 2018, https://webusers.imj-prg.fr/~marco.robalo/GDT-THH-2018/NotesGeoffroyTate.pdf

[NS19] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2019), no. 2, pp. 203–409, preprint arXiv:1707.01799v2

[Toë20] B. Toën, *Le problème de la schématisation de Grothendieck revisité*, 2020, arXiv:1911.05509

[Yua19] A. Yuan, *Integral models for spaces via the higher Frobenius*, 2019, arXiv:1910.00999