

Integral homotopy theory (after Allen Yuan)

A. Yuan, "Integral models for spaces via the higher Frobenius", arXiv

Thm (Whitehead) If X, Y are simply connected spaces and $f: X \rightarrow Y$ induces an iso $H_*(X, \mathbb{Z}) \xrightarrow{\sim} H_*(Y, \mathbb{Z})$, then f is an equivalence.

Dcf. Let X be a s.c. space.

- X is rational if $\pi_i(X)$ are \mathbb{Q} -vector spaces
(of finite type) (finite-dim)
- X is p-complete (of finite type) if $\pi_i(X)$ are p-complete (top. finitely generated)

\exists unique morphism $X \rightarrow X_{\mathbb{Q}}$ where $X_{\mathbb{Q}}$ is rational
and $X \rightarrow X_{\mathbb{Q}}$ induces an iso on $H_*(-, \mathbb{Q})$

$\xrightarrow{\quad}$ $X \rightarrow X_p^1$ where X_p^1 is p-complete
and $X \rightarrow X_p^1$ is an iso on $H_*(-, \mathbb{F}_p)$.

Thm (Quillen 69, Sullivan 77)

The functor $\{\text{spaces}\}^{op} \rightarrow \text{CAlg}_{\mathbb{Q}}$
 $X \mapsto C^*(X, \mathbb{Q})$

is fully faithful on s.c. rational spaces of finite type.

important

Notation - \mathcal{S} n-cat of spaces
• $\text{CAlg}_{\mathbb{Q}}$ n-category
of Eo-R-algebras.

↳ can remove this by replacing
 $C^*(X, \mathbb{Q})$ by $C_*(X, \mathbb{Q})$

Rmk. $\Rightarrow X \simeq \text{Map}(*, X) \simeq \text{Map}_{\text{CAlg}_{\mathbb{Q}}} (C^*(X, \mathbb{Q}), \mathbb{Q})$

Thm (Mandell 2001)

The functor $\mathcal{S}^{op} \rightarrow \text{CAlg}_{\mathbb{F}_p}$
 $X \mapsto C^*(X, \mathbb{F}_p)$

is fully faithful on s.c. p-complete spaces of finite type.

($\Rightarrow X \simeq \text{Map}_{\text{CAlg}_{\mathbb{F}_p}} (C^*(X, \mathbb{F}_p), \mathbb{F}_p)$.)

Thm (Sullivan's arithmetic square) If X is s.c. space s.t. $H^n(X, \mathbb{F}_p)$ are finite for all n, p, then there is a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^1 \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p^1)_{\mathbb{Q}} \end{array}$$

If X is a finite space, then $C^*(X, \mathbb{R}) \simeq C^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and

$$\begin{aligned} C^*(X, \mathbb{Z}) &\longrightarrow \prod_p C^*(X, \mathbb{Z}_p) \\ \downarrow & \quad \downarrow \\ C^*(X, \mathbb{Q}) &\longrightarrow \left(\prod_p C^*(X, \mathbb{Z}_p) \right) \otimes \mathbb{Q} \end{aligned}$$

If Mandell's theorem was true for \mathbb{F}_p instead of $\widehat{\mathbb{F}_p}$, then we would deduce that $C^*(-, \mathbb{Z}) : \{\text{finite spaces}\}^{\text{op}} \rightarrow \text{Alg}_{\mathbb{Z}}$ is fully faithful.

Then (Mandell, 2006): $C^*(-, \mathbb{Z}) : \{\text{s.c. spaces of f.t.}\}^{\text{op}} \rightarrow \text{Alg}_{\mathbb{Z}}$ is faithful on homotopy categories.

Goal: obtain a fully faithful embedding.

$$\begin{aligned} &\text{(} X \text{ s.c. } p\text{-complete)} \\ &\text{Map}_{\text{Alg}_{\mathbb{F}_p}}(C^*(X, \mathbb{F}_p), \mathbb{F}_p) \simeq \text{Map}_{\text{Alg}_{\widehat{\mathbb{F}_p}}}((C^*(X, \mathbb{F}_p), \widehat{\mathbb{F}_p})^{h\mathbb{Z}}) \xrightarrow[\text{by Fröbeinus}]{{\mathbb{Z}} \cong \widehat{\mathbb{F}_p}} \\ &\simeq \text{Map}_{\text{Alg}_{\widehat{\mathbb{F}_p}}}((C^*(X, \widehat{\mathbb{F}_p}), \widehat{\mathbb{F}_p})^{h\mathbb{Z}}) \\ &= X^{h\mathbb{Z}} \simeq \text{Map}_{\widetilde{\mathcal{S}'}}(\widetilde{B\mathbb{Z}}, X) = \mathcal{L}X \text{ free loop space} \end{aligned}$$

S' acts on $\mathcal{L}X$ and $X = (\mathcal{L}X)^{hS'}$.

Classically: an action of S' on an ordinary category \mathcal{C} is an automorphism of id \mathcal{C} $\xrightarrow[B\mathbb{Z}]{S'} \text{Aut}(\mathcal{C})$
 $\Rightarrow \exists$ an action of S' on $(\text{Alg}_{\widehat{\mathbb{F}_p}})^{\text{op}}$ by Fröbeinus.

More generally, Fröbeinus defines an action of the monoidal category BIN on $\text{Alg}_{\widehat{\mathbb{F}_p}}^{\heartsuit}$.

However: Fröbeinus on E_{∞} - \mathbb{F}_p -algebra is not \mathbb{F}_p -linear.

Nikolaus-Scholze Fröbeinus If A is an E_{∞} -ring, then there is an E_{∞} -ring

$$\text{map } \varphi_A : A \longrightarrow A^{\text{tcf}_p} \quad \begin{cases} E \text{ spectrum with } \mathbb{F}_p\text{-action} \\ "x \mapsto x^p" \end{cases}$$

↑ Tate construction/
Tate fixed points

If A is discrete, $\pi_0(A^{\text{tcf}_p}) = A/p$ and $\pi_0(\varphi_A)$ is the usual Fröbeinus.

Then (legal conjecture for \mathbb{F}_p , Lin, Gunnarsson) $\mathbb{F}^{\text{tcf}_p} \simeq \mathbb{F}_p^{\text{tcf}}$

$\Rightarrow \varphi_{S_p}$ is an endomorphism of S_p^1 (in fact it's the identity).

Yuan's results (Thm A, B, C) $\hookrightarrow p\text{-complete } E_\infty\text{-nug}$

\exists full subcategory $CAlg_p^F \subset CAlg_p$ where φ is an endomorphism

$$\Phi: BN \longrightarrow \text{End}(CAlg_p^F)$$

Thm A Φ can be promoted to a monoidal functor, i.e., it defines an action of BN on $CAlg_p^F$.

Def. $CAlg_p^{\text{perf}} \subset CAlg_p^F$ full subcategory where φ is an automorphism.

Ex: X finite space, $(S_p^1)^X$ is a p -perfect E_∞ -nug.

So $S^1 \otimes CAlg_p^{\text{perf}}$ via Frobenius.

Rmk. S^1 -action on an ∞ -category is much more than an endomorphism of id: $CP^\infty \cong BS^1 \longrightarrow \text{Cat}_\infty$

$$S^2 \xrightarrow{\text{U}} \text{Cat}_\infty \xrightarrow{\text{action of id}}$$

Def. $CAlg_p^{\varphi=1} = (CAlg_p^{\text{perf}})^{hs^1}$

Thm B.

$$\begin{array}{ccc} \{ \text{s.c. finite } p\text{-complete spaces} \}^{\text{op}} & \xrightarrow{\quad X \mapsto (S_p^1)^X \quad} & CAlg_p \\ \xrightarrow{\quad \text{cannot relax} \quad} & \exists & \xrightarrow{\quad \text{fully faithful} \quad} CAlg_p^{\varphi=1} \\ \text{to "finite type".} \\ \text{e.g. } (S_p^1)^{\text{op}} \notin CAlg_p^F \end{array}$$

Def. (Frobenius-fixed E_∞ -nug).

$$CAlg^{\varphi=1} \longrightarrow \prod_p CAlg_p^{\varphi=1}.$$

$$\downarrow \text{PB} \qquad \downarrow \text{forget}$$

$$CAlg \longrightarrow \prod_p CAlg_p.$$

Thm C

$$\begin{array}{ccc} \{ \text{s.c. finite spaces} \}^{\text{op}} & \xrightarrow{\quad X \mapsto S^X \quad} & CAlg \\ \xrightarrow{\quad \text{fully faithful} \quad} & \exists & \xrightarrow{\quad P \quad} CAlg^{\varphi=1} \end{array}$$

More on Thm A.

I. Norm functors in equivariant stable homotopy theory.

Monoidal category Q : objects are finite groups, morphisms $G \xleftarrow{H} K$.

Thm A1 \exists action of Q on $CAlg(S_p)$ which refines E_∞ -Frobenius.

$$G_p \rightsquigarrow (-)^{t_{G_p}}$$

$\star \leftarrow \pi \hookrightarrow G_p \rightsquigarrow id \rightarrow (-)^{t_{G_p}}$ Frobenius.

II. Partial K-theory: non-group-complete K-theory: $K(\mathbb{C}) \cong K^{\text{perf}}(\mathbb{C})^{\otimes p}$

$$\begin{aligned} Q(\text{Vect}_{\mathbb{F}_p}) &\subset Q && \xrightarrow{\text{localization of } Q(\text{Vect}_{\mathbb{F}_p})} \\ &\rightsquigarrow \exists \text{ action of } BK^{\text{perf}}(\mathbb{F}_p) \text{ on } \mathcal{CAlg}_p^F \\ \text{Thm A2} \quad K^{\text{perf}}(\mathbb{F}_p) &\xrightarrow{\text{rk}} \mathbb{N} \text{ is an } \mathbb{F}_p\text{-homology isomorphism.} \\ (\text{compare: Quillen: } K(\mathbb{F}_p) &\xrightarrow{\text{rk}} \mathbb{Z} \xrightarrow{\quad\quad\quad}) \\ &\rightsquigarrow B\mathbb{N} \cong \mathcal{CAlg}_p^F. \end{aligned}$$