Norm Maps, The Tate Construction, and the Tate Diagonal

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Part I

Norm Maps and Ambidexterity

Integrals of families of maps

Definition

Let X be an ∞-groupoid, $\mathcal{C}$ an ∞-category with X-indexed limits and colimits. Then an *X-measure* on $\mathcal{C}$ is a natural transformation $\mu : \Delta_* \to \Delta_!$, where $\Delta^* : \mathcal{C} \to \mathrm{Fun}(X, \mathcal{C})$ is the diagonal functor sending an object c to the constant diagram at c and $\Delta_!$ and Δ_* denote its left and right adjoint respectively.

Construction

Let $f \colon X \to \mathcal{C}(c, d)$ be a family of maps. Then this corresponds under adjunction to a map $f \colon \Delta^* c \to \Delta^* d$ in $\mathrm{Fun}(X, \mathcal{C})$. Let μ be an X-measure on $\mathcal{C}$. Then we define the integral of f with respect to μ to be

$$\int_X f \mathrm{d}\mu : c \xrightarrow{\eta} \Delta_* \Delta^* c \xrightarrow{\Delta_* f} \Delta_* \Delta^* d \xrightarrow{\mu} \Delta_! \Delta^* d \xrightarrow{\varepsilon} d$$

The norm $\mathrm{Nm}_{\boldsymbol{\mu}}$ for a parametric measure

Definition

A *parametric X-measure* $\boldsymbol{\mu}$ on $\mathcal{C}$ is a functorial family of $X(x, y)$-measures μ_{xy} on $\mathcal{C}$.

Construction

Let $\mathcal{L} : X \to \mathcal{C}$ be a functor and let $\boldsymbol{\mu}$ be a parametric X-measure on $\mathcal{C}$. Then for each pair $x, y \in X$, we have an associated map $\mathcal{L}_{xy} : X(x, y) \to \mathcal{C}(\mathcal{L}_x, \mathcal{L}_y)$. Taking integrals pointwise, we obtain a natural family of maps $\int_{X(x,y)} \mathcal{L}_{xy} \, \mathrm{d}\mu_{x,y} : \mathcal{L}_x \to \mathcal{L}_y$. Then we obtain a map functorial in L:

$$\mathrm{Nm}_{\boldsymbol{\mu}} : \operatorname*{colim}_{x \in X} \mathcal{L}_x \to \lim_{xinX} \mathcal{L}_x.$$

called the *norm map* relative to $\boldsymbol{\mu}$.

Recursive Norm Maps

Definition

Let $\mathcal{C}$ be an ∞-category admitting enough limits and colimits. Then the map $\Delta^* : \mathcal{C} \to Fun(*, \mathcal{C})$ is an equivalence, and the identity natural transformation $\Delta_* \to \Delta_!$ defines a $*$-measure on $\mathcal{C}$ called the HL measure.

Recursion

Let X be an n-truncated ∞-groupoid. Suppose that the HL measure on $\mathcal{C}$ has been constructed and is functorial for all path spaces $X(x, y)$ with $x, y \in X$. Then these measures give a parametric X-measure $\boldsymbol{\mu}_{HL}$ on $\mathcal{C}$, and therefore a norm map

$$\mathrm{Nm}_{HL} : \Delta_{X!} \to \Delta_{X*}.$$

If Nm_{HL} is invertible, define the HL X-measure μ_{HL} on $\mathcal{C}$ to be Nm_{HL}^{-1}.

Ambidexterity

Theorem (Hopkins-Lurie, Ambidexterity)

The above can all be made coherently functorial and natural using the theory of ambidexterity in Beck-Chevalley fibrations. In particular, as long as we assume $\mathcal{C}$ *is tensored and cotensored over* $\mathcal{S}$, *there is a biCartesian fibration* $H^{\mathcal{C}} \to \mathcal{S}$ *satisfying the Beck-Chevalley condition and such that the fibre over a space* X *is* $H^{\mathcal{C}}_X = \mathrm{Fun}(X, \mathcal{C})$. *There are two classes of maps defined by mutual induction in* $\mathcal{S}$ *called weakly* $\mathcal{C}$*-ambidextrous and* $\mathcal{C}$*-ambidextrous maps, and a map of spaces is (weakly)* $\mathcal{C}$*-ambidextrous if and only if all of its fibres are. Whenever a map* f *is weakly ambidextrous, there is a norm map*

$$\mathrm{Nm}_f : f_! \to f_*,$$

and f *is ambidextrous if and only if this map is an equivalence.*

Ambidexterity

Remark

In the language of integrals and measures, the parametric X-measure μ_{HL} on $\mathcal{C}$ (and therefore the norm map Nm_{HL}) is defined precisely when the map $X \to *$ is weakly ambidextrous, and it is an equivalence precisely when $X \to *$ is ambidextrous.

Examples

- If X is empty, then the norm map is the map from the initial object to the terminal object. It follows that X is $\mathcal{C}$-ambidextrous if and only if $\mathcal{C}$ is pointed.
- If X is a set, then the norm map

$$\mathrm{Nm}_X : \coprod_{x \in X} F(x) \to \prod_{x \in X} F(x)$$

  exists if $\mathcal{C}$ is pointed and is given by the diagonal matrix with the identity down the diagonal. Note that $\mathcal{C}$ is preadditive if and only if every finite set X is $\mathcal{C}$-ambidextrous.
- For G a finite group, the norm of a complex G-representation V is the map $V_G \to V^G$ sending an orbit to the sum of its elements divided by the order of G.

The Tate construction

Definition

Given a weakly $\mathcal{C}$-ambidextrous ∞-groupoid X, we define the *Tate Construction* to be the functor $(-)^{tX}: \mathrm{Fun}(X, \mathcal{C}) \to \mathcal{C}$ defined by the rule $F \mapsto F^{tX}$ where

$$F^{tX} := \mathrm{cofib}(\operatorname*{colim}_X F \xrightarrow{\mathrm{Nm}_X} \lim_X F).$$

More generally, given a weakly $\mathcal{C}$-ambidextrous map $f: X \to Y$ of ∞-groupoids, there is also a relative Tate construction obtained by taking the cofibre of the relative norm map map

$$F^{tf} = \mathrm{cofib}(f_! \xrightarrow{\mathrm{Nm}_f} f_*)$$

When $X = BG$, by abuse of notation, we denote F^{tBG} by F^{tG}.

Part II

Multiplicativity of the Tate Construction

Multiplicativity of Tate

Theorem (Nikolaus-Scholze)

Let G be a finite group, and consider the category $\mathbf{Sp}^{BG}$ of representations of G in spectra. Then the Tate construction $\mathbf{Sp}^{BG} \to \mathbf{Sp}$ is lax symmetric monoidal and equipped with a lax symmetric monoidal refinement of the natural transformation $\bullet^{hG} \to \bullet^{tG}$. Moreover, these data are unique up to a contractible space of choices.

Corollary

Given an ∞-operad $\mathcal{O}$, homotopy G-invariants and the Tate construction extend to functors $\mathrm{Alg}_{\mathcal{O}}(\mathbf{Sp}^{BG}) \to \mathrm{Alg}_{\mathcal{O}}(\mathbf{Sp})$, and the natural transformation connecting them extends to a natural transformation between the extensions.

Proof Strategy

- Carefully re-prove the existence of Verdier quotients by exhibiting explicit constructions. This helps to understand the localization map $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ for $\mathcal{D} \subset \mathcal{C}$ stable.
- Demonstrate further that when $\mathcal{C}$ is a stably symmetric monoidal stable ∞-category and $\mathcal{D}$ is a $\otimes$ ideal, the category $\mathcal{C}/\mathcal{D}$ has an induced stably symmetric monoidal structure, and the functor $\mathcal{C} \to \mathcal{C}/\mathcal{D}$ is exact symmetric-monoidal.
- Moreover, show that this functor is universal among exact lax symmetric monoidal functors killing $\mathcal{D}$. This universality property gives a way to approximate a lax symmetric monoidal functor $\mathcal{C} \to \mathcal{E}$ to one killing $\mathcal{D}$ when taking maps into a presentably symmetric monoidal stable ∞-category $\mathcal{E}$.
- When G is a finite group, show that the category of spectral G-representations killed by $\bullet^{tG}$ is a $\otimes$-ideal.
- Exhibit $\bullet^{tG}$ as the image of $\bullet^{hG}$ under the adjunction noted before and the natural transformation $\bullet^{hg} \to \bullet^{tG}$ as the unit of the adjunction.

Verdier Quotients

Theorem

Let $\mathcal{D} \subset \mathcal{C}$ be a full stable subcategory of a small stable category $\mathcal{C}$. Then the Verdier quotient $\mathcal{C}/\mathcal{D}$ is stable and can be constructed as an explicit localization of $\mathcal{C}$ at the set of arrows W in $\mathcal{C}$ whose cofibre lies in $\mathcal{D}$. Moreover, by the universal property of localizations, we have a fully faithful functor $\mathcal{C}/\mathcal{D} \to \mathrm{Psh}(\mathcal{C})$, and it follows by an explicit calculation that this functor factors through an exact functor $\mathcal{C}/\mathcal{D} \to \mathrm{Ind}(\mathcal{C})$.

Proof Sketch

Proof.

The calculation of the spaces $\mathcal{C}[W^{-1}](x, y)$ can be performed by appealing to the Yoneda lemma together with the adjoint functor theorem. By universal property, $\mathcal{C}[W^{-1}](x, -) = L(\mathcal{C}(x, -)$, where L is the localization functor left adjoint to the fully faithful inclusion

$$\iota : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S}).$$

So we would like to compute the localization functor L. It is a straightforward exercise to verify that we can take L to be the functor

$$F \mapsto \operatorname*{colim}_{\alpha \in \mathcal{D}_{/x}} F(\mathrm{cofib}(\alpha)).$$

Note that $\mathcal{D}_{/x}$ is filtered because $\mathcal{D}$ is stable, so ι factors through $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$ (mutatis mutandis for the opposites). We leave it as an exercise to check that $\mathcal{C}[W^{-1}]$ is stable. □

Verdier Quotients, Presentable Desiderata

Corollary

Let $\mathcal{E}$ be a presentable stable ∞-category. Then in the situation above, the fully faithful inclusion functor $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subseteq \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}, \mathcal{E})$ *admits a left adjoint obtained by the composite*

$$\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Pr}^{\mathsf{L}}(\mathsf{Ind}(\mathcal{C}), \mathcal{E}) \to \mathrm{Pr}^{\mathsf{L}}(\mathsf{Ind}(\mathcal{C}/\mathcal{D}), \mathcal{E}) \simeq \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}/\mathcal{D}, \mathcal{E}),$$

where the middle functor $\mathsf{Ind}(\mathcal{C}) \to \mathsf{Ind}(\mathcal{C}/\mathcal{D})$ *is induced by the exact functor* $\mathcal{C}/\mathcal{D} \to \mathsf{Ind}(\mathcal{C})$.

Proof.

Use the explicit description from the proof of the theorem to demonstrate the adjunction. □

Verdier Quotients by $\otimes$-Ideals

Theorem

Let $\mathcal{C}$ be a small stably symmetric monoidal stable ∞-category, and let $\mathcal{D}$ be a $\otimes$-ideal. Then the Verdier quotient map $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ admits a symmetric monoidal refinement that equips $\mathcal{C}/\mathcal{D}$ with a compatible stably symmetric monoidal structure. Moreover, this functor has the following universal property:
If $\mathcal{E}$ is any other stably symmetric monoidal stable ∞-category, then precomposition with the symmetric monoidal Verdier projection π above induces a fully faithful functor

$$\mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subseteq \mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}, \mathcal{E})$$

with essential image given by those lax symmetric monoidal exact functors that kill $\mathcal{D}$.

Proof.

This result is a direct application of a theorem of Hinich combined with the universal property of Verdier quotients. □

Symmetric Monoidal Verdier Quotients, Presentable Desiderata

Corollary

If $\mathcal{E}$ is presentably symmetric monoidal stable, then the fully faithful inclusion

$$\mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \subseteq \mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}, \mathcal{E})$$

admits a left adjoint reflector given by

$$\mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}^{\mathbf{pres}}_{\mathrm{lax}}(\mathsf{Ind}(\mathcal{C}), \mathcal{E}) \to \mathsf{Fun}^{\mathbf{pres}}_{\mathrm{lax}}(\mathsf{Ind}(\mathcal{C}/\mathcal{D}), \mathcal{E}) \to \mathsf{Fun}^{\mathbf{ex}}_{\mathrm{lax}}(\mathcal{C}/\mathcal{D}, \mathcal{E}).$$

Proof.

It suffices to check that the induced functor $\mathsf{Ind}(\mathcal{C}/\mathcal{D}) \to \mathsf{Ind}(\mathcal{C})$ is lax symmetric monoidal, but this is the case as it is right adjoint to the presentable symmetric monoidal projection functor $\mathsf{Ind}(\pi) : \mathsf{Ind}(\mathcal{C}) \to \mathsf{Ind}(\mathcal{C}/\mathcal{D})$. The result now follows by the previous corollary. □

First Lemma

Lemma

Let G be a finite group, and let $\mathbf{Sp}^{BG}_{\mathrm{ind}} \subset \mathbf{Sp}^{BG}$ denote the full stable subcategory generated by the induced G-representations. Then this subcategory is a $\otimes$-ideal on which the Tate construction vanishes.

Proof Sketch

Proof.

Consider the map from the terminal category to BG. Then the induced and coinduced representations are given by left and right Kan extensions along this map. Using the conical formula for left and right Kan extensions, we see that the Norm map induced by this map is an equivalence. By naturality of the norm map, we see that the norm map $x_{hG} \to x^{hG}$ is an equivalence for all induced representations x of G, so the Tate construction vanishes on them. But the kernel of the Tate construction is stable, so it also vanishes on the full stable subcategory generated by them.
To see that $\mathbf{Sp}^{BG}_{\mathrm{ind}}$ is a $\otimes$-ideal, notice that for any G-representation x, the category of $y \in \mathbf{Sp}^{BG}_{ind}$ such that $x \otimes y \in \mathbf{Sp}^{BG}_{ind}$ is stable, so we can check on induced representations. But in this case, we can take a non-equivariant inclusion of a summand $z \to y$ and give a non-equivariant map $x \otimes z \to x \otimes y$ that extends by adjunction to an equivalence. But this is a induced G-representation and therefore induced. □

Second Lemma

Lemma

The subcategory $\mathbf{Sp}^{BG}_{\mathrm{ind}} \subset \mathbf{Sp}^{BG}$ under canonically filtered colimits, that is, the canonical map

$$\operatorname*{colim}_{d \to x \in \mathbf{Sp}^{BG}_{\mathrm{ind}/x}} d \to x$$

is an equivalence for all G-representations x.
Finally, for any G-representation x, the canonical map

$$\operatorname*{colim}_{\alpha \in \mathbf{Sp}^{BG}_{\mathrm{ind}/x}} (\mathrm{cofib}(\alpha))^{hG} \to \operatorname*{colim}_{\alpha \in \mathbf{Sp}^{BG}_{\mathrm{ind}/x}} (\mathrm{cofib}(\alpha))^{tG}$$

is an equivalence.

Proof Sketch

Proof.

The first claim is proved by an appeal to homotopy groups, since the slice category is filtered. By taking shifts, it is enough to prove it in the case of π_0, so we can check injectivity and surjectivity. This is left as an exercise.
The second claim follows from the first using the fibre sequence for the Tate construction pointwise. The coinvariants term vanishes because coinvariants commute with all colimits. □

Proof of Main Theorem

Proof.

Let $\mathcal{C} = \mathbf{Sp}^{BG}$ and $\mathcal{D} = \mathbf{Sp}^{BG}_{\mathrm{ind}}$. Then the homotopy invariants functor is a lax symmetric monoidal exact functor $\mathcal{C} \to \mathbf{Sp}$. Denote it by F. Since $\mathbf{Sp}$ is presentable symmetric monoidal stable, there exists a universal lax symmetric monoidal functor $H : \mathcal{C}/\mathcal{D} \to \mathbf{Sp}$ such that the restriction to $\mathcal{C}$ receives a lax symmetric monoidal transformation $\eta : F \to H|_{\mathcal{C}}$. Then we must show that H is the Tate construction and η is the cofibre of the norm map. By the compatibility of the localizations, it suffices to check this without taking into account monoidal structures. Since the Tate construction is exact, and vanishes on $\mathcal{D}$, and receives the cofibre of the norm map from F, there exists a universal map $\eta : H \to \cdot^{tG}$. But the explicit computations of the second lemma above together with the formula for the localization tell us precisely that η is an equivalence. □

The C_p-Tate Tensor Power

Definition

For every prime p, we have a functor $\zeta_p : \mathbf{Sp} \to \mathbf{Sp}^{BC_p}$ sending a spectrum to its p tensor power with C_p-action given by cyclic permutation of the factors. Composing this functor with the Tate construction, we obtain a functor $T_p : \mathbf{Sp} \to \mathbf{Sp}$.

Lemma

The functor T_p is exact.

Proof.

It suffices to show that T_p preserves extensions. Nikolaus and Scholze first prove it preserves sums, then for general fibre sequences, they appeal to the associated filtration of the middle term. □

What filtration?

Spectral Yoneda Trick

Lemma

If $F : \mathbf{Sp} \to \mathbf{Sp}$ is any exact functor, then a natural transformation $\mathrm{id}_{\mathbf{Sp}} \to F$ is determined up to contractible ambiguity by a choice of map $\mathbb{S} \to F(\mathbb{S})$.

Proof.

We have

$$Fun^{\mathbf{ex}}(\mathbf{Sp}, \mathbf{Sp}) \simeq \mathrm{Fun}^{\mathbf{lex}}(\mathbf{Sp}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathbf{Sp}, \mathcal{S}),$$

with the first map being an equivalence by composition with Ω^∞, and the second map being a full inclusion. But the sphere spectrum corepresents the functor Ω^∞, so by Yoneda, we have that the space of natural transformations

$$\mathrm{Map}(\mathrm{id}_{\mathbf{Sp}}, F) \simeq \Omega^\infty F(\mathbb{S}) \simeq \mathbf{Sp}(\mathbb{S}, F(\mathbb{S})).$$

□

The Tate Diagonal

Definition

We define the Tate diagonal to be the natural transformation $\mathrm{id}_{\mathbf{Sp}} \to T_p$ corresponding to the map of spectra.

$$\mathbb{S} \to \mathbb{S}^{hC_p} \to T_p(\mathbb{S}).$$

Note

It can be shown that T_p is a lax symmetric monoidal exact functor. We showed already that the Tate construction is exact lax symmetric monoidal, but it would remain to show that the functor $\mathcal{C} \to \mathcal{C}^{BC_p}$ sending $x \mapsto x^{\otimes p}$ with the action given by cyclic permutations is lax symmetric monoidal. This is not proven in the paper (in fact the construction of this functor is not given!). It then follows by a theorem of Nikolaus that the identity functor is the initial exact lax symmetric monoiodal functor $\mathbf{Sp} \to \mathbf{Sp}$, and the universal lax symmetric monoidal transformation to T_p is exactly the Tate diagonal.