

Equivariant stable homotopy theory

G finite group \rightsquigarrow sgm.-monoidal ∞ -category Sp_G of genuine G -spectra

Functoriality in G

$$\begin{array}{ll} \begin{cases} H \xrightarrow{f} G \\ f^*: \mathrm{Sp}_G \rightarrow \mathrm{Sp}_H \end{cases} & \begin{cases} 1) \quad H \subset G: \quad \mathrm{rel}_H^G: \mathrm{Sp}_G \rightarrow \mathrm{Sp}_H \text{ has left & right adjoints (ind & coind)} \\ 2) \quad G \rightarrow G/N: \quad \mathrm{triv}_{G/N}^G: \mathrm{Sp}_{G/N} \rightarrow \mathrm{Sp}_G \text{ has right adjoint } (-)^N \end{cases} \\ \begin{cases} H \xleftarrow{f} G \\ f_*: \mathrm{Sp}_H \rightarrow \mathrm{Sp}_G \end{cases} & \begin{cases} 3) \quad H \subset G: \quad N_H^G: \mathrm{Sp}_H \rightarrow \mathrm{Sp}_G \text{ (Hill-Hopkins-Ravenel norm)} \\ 4) \quad G \rightarrow G/N: \quad \Phi^N: \mathrm{Sp}_G \rightarrow \mathrm{Sp}_{G/N} \text{ geometric fixed points} \end{cases} \end{array}$$

$$\begin{array}{ccc} \text{Goals:} & \text{Construct} & \mathrm{Span}(\mathrm{FinGpd}) \longrightarrow \mathrm{CAlg}(\mathrm{Cat}_\infty^\text{soft}) \\ & \xrightarrow{\text{funct } \pi_0 \text{ & } \pi_1} & X \simeq \coprod_i BG_i \mapsto \mathrm{Sp}_X \simeq \prod_i \mathrm{Sp}_{G_i} \\ & \xleftarrow{f} Y \xrightarrow{g} Z & \xleftarrow{f} Y \xrightarrow{g} Z \mapsto \mathrm{Sp}_X \xrightarrow{f^*} \mathrm{Sp}_Y \xrightarrow{\partial \otimes} \mathrm{Sp}_Z. \end{array}$$

Definition of Sp_X $X \in \mathrm{FinGpd}$.

$\mathrm{Fin} = \text{category of finite sets}, \mathrm{Fin}_X = \mathrm{Fin}(X, \mathrm{Fin}) \subset \mathrm{FinGpd}/_X$
full subcategory on
0-truncated maps $Y \rightarrow X$

Elmendorff's Theorem

$$(\text{G-CW-complexes})[\text{htpy equiv}] \simeq P_\Sigma(\mathrm{Fin}_{BG})$$

↗ sifted completion
 ≡ preserves filtered colimits

$$\begin{array}{ll} \text{Def. } \mathfrak{S}_X = P_\Sigma(\mathrm{Fin}_X) & \text{"X-spectra"} \\ \mathrm{Sp}_X = \lim (\dots \rightarrow (\mathfrak{S}_X)_* \xrightarrow{\Omega^R} (\mathfrak{S}_X)_* \xrightarrow{\Omega^R} (\mathfrak{S}_X)_*) & \\ \Sigma^\infty \uparrow \downarrow \Omega^\infty & \Omega^R = \mathrm{Hom}(S^R, -), \quad S^R = \text{regular representation sphere} \\ (\mathfrak{S}_X)_* & \text{real } \mathbb{R}[G] \end{array}$$

Fact: $\Sigma^\infty: (\mathfrak{S}_X)_* \rightarrow \mathrm{Sp}_X$ has a sgm.-monoidal structure with the following universal property: Let K be a collection of simplicial sets containing the filtered ones. If $C \in \mathrm{CAlg}(\mathrm{Cat}_\infty^K)$, the functor

$$\mathrm{Fun}^{S^R, K}(\mathrm{Sp}_X, C) \longrightarrow \mathrm{Fun}^{S^R, K}((\mathfrak{S}_X)_*, C)$$

is fully faithful and $F: (\mathfrak{S}_X)_* \rightarrow C$ is in the image iff $F(S^R)$ is \otimes -invertible in C .

follows formally from: $(S_x)_*$ is compactly generated and Sp is compact.
(Robalo) $\text{on } (\mathbb{S}^1)^{\wedge 3}$, the cycle permutation is homotopic to the identity.

Functoriality of $X \mapsto \text{Sp}_X$

$$f: Y \rightarrow X \text{ in } \text{FinGpd}, \Rightarrow f^*: \text{Fin}_Y \rightleftarrows \text{Fin}_X : f_*$$

$$\text{Beck-Chevalley property: } (f_* A)_x = \lim_{y \in f^{-1}(x)} A_y.$$

$$\begin{array}{ccc} Y' & \xrightarrow{k} & Y \\ \downarrow j & & \downarrow f \\ X & \xrightarrow{h} & X \end{array} \text{ cartesian} \Rightarrow f^* h_* \xrightarrow{\sim} h_* g^*$$

$$\Rightarrow \exists \text{ Span}(\text{FinGpd}) \longrightarrow \text{Cat}_{\infty}^{\text{lex}} \quad (\text{unfurly}) \quad (1)$$

$$\begin{array}{ccc} X & \longmapsto & \text{Fin}_X \\ f \swarrow \quad \searrow g & & f^* \end{array}$$

$$\begin{aligned} \text{Dagger on } P_{\Sigma}: P_{\Sigma}: \text{Cat}_{\infty} &\longrightarrow \text{Cat}_{\infty}^{\text{sift}} && \text{sifted completion} \\ P_{\Sigma}(C \times D) &= P_{\Sigma}(C) \times P_{\Sigma}(D) \\ \Rightarrow P_{\Sigma}: \text{CAlg}(\text{Cat}_{\infty}) &\longrightarrow \text{CAlg}(\text{Cat}_{\infty}^{\text{sift}}) && (\text{Day convolution}) \end{aligned}$$

Suppose C has finite \amalg and a final object.

Let $C_+ \subset C_*$ full subcategory on $X \amalg *$

$$\Rightarrow P_{\Sigma}(C)_* = P_{\Sigma}(C_+) \quad (\text{Kuroda's})$$

If C has a sym. monoidal structure that distributes over \amalg , then this is sym. monoidal equivalence.

$$\text{Hence } (S_X)_* = P_{\Sigma}(\text{Fin}_{X+})$$

Remarks:

- $\text{Fin}_{X+} = \text{Fin}_{X*}$ subcategory on $A \xleftarrow{f} B \xrightarrow{g} C$ $B \hookrightarrow A$ mono.
- $\text{Fin}_X \subset \text{Fin}_{X+} \subset \text{Span}(\text{Fin}_X)$
- $A \mapsto A_+ \mapsto A$
- $A_+ \xrightarrow{f} B_+ \mapsto A \xleftarrow{f^*(B)} B$

$$\text{Span}: \text{Cat}_{\infty}^{\text{lex}} \longrightarrow \text{CAlg}(\text{Cat}_{\infty}) \quad \otimes \text{ on } \text{Span}(C) \cong \times \text{ in } C.$$

$$\text{Span} \circ (1): \begin{array}{ccc} \text{Span}(\text{FinGpd}) & \longrightarrow & \text{CAlg}(\text{Cat}_{\infty}) \\ X & \longmapsto & \text{Span}(\text{Fin}_X) \end{array} \quad \begin{array}{l} \text{Span}(f^*) = f^* \\ \text{Span}(g_*) = g \otimes \end{array} \quad (2)$$

For $f: Y \rightarrow X$, f^* and f_* preserve Fin_+
(because f^* and f_* preserve monomorphisms)

$$\Rightarrow \text{we get } \begin{aligned} \text{Span}(\text{FinGpd}) &\longrightarrow \text{Cat}_{\infty}(\text{CAlg}) \\ X &\longmapsto \text{Fin}_{X+}, f^*, g_* \\ \text{as a subfunctor of (2)} \end{aligned} \quad (3)$$

- Exercise:
- $\nabla: X \amalg X \rightarrow X$, $\nabla_{\otimes}: \text{Fin}_{(X \amalg X)_+} \cong \text{Fin}_X \times \text{Fin}_X \rightarrow \text{Fin}_X$
is the smash product functor
 - $p: * \rightarrow BG$, $p_{\otimes}: \text{Fin}_+ \rightarrow \text{Fin}_{BG+}$, $A \mapsto A^{AG}$

$$P_2 \circ (3): \begin{aligned} \text{Span}(\text{FinGpd}) &\longrightarrow \text{Cat}_{\infty}(\text{CAlg}^{\text{aff}}) \\ X &\longmapsto P_2(\text{Fin}_X) \cong (\mathbb{S}_X)_* \end{aligned} \quad (4)$$

Stabilization. $\text{OCat}_{\infty} = \infty$ -categories with a collection of objects.

$$\begin{array}{c} \downarrow \\ \text{Cat}_{\infty} \end{array} \xleftarrow{\text{cocartesian fibration in posets}}$$

The functor $\text{Cat}_{\infty}(\text{CAlg}_{\infty})^{\text{aff}} \rightarrow \text{Cat}_{\infty}(\text{OCat}_{\infty}^{\text{aff}}) \subset \text{Cat}_{\infty}(C, \text{Pic}(C))$
has a left adjoint sending $(\mathbb{S}_X)_*, \{S^1\} \mapsto \text{Sp}_X$.

Lemma: For every $p: Y \rightarrow X$ in FinGpd , the composition

$$(\mathbb{S}_Y)_* \xrightarrow{P_{\otimes}} (\mathbb{S}_X)_* \xrightarrow{\Sigma^{\infty}} \text{Sp}_X$$

sends S^1 to an invertible object.

$$\text{Pf: } Y \xrightarrow{q} Z \xrightarrow{r} X \quad q \text{ is 1-connected} \\ r \text{ is 0-connected}$$

q_{\otimes} preserves colimits since $q_*: \text{Fin}_Y \rightarrow \text{Fin}_Z$ preserves \amalg

$$\Rightarrow q_{\otimes}(S^1) = S^1$$

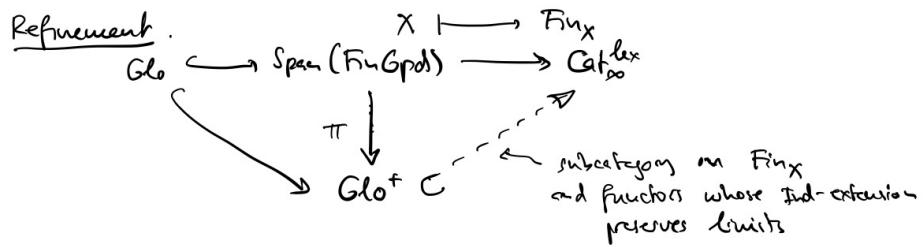
wLOG p is 0-truncated. wLOG: $p: BH \rightarrow BG$ where $H \subset G$

$$\text{Then } p_{\otimes}(S^1) = \bigwedge_{G/H} S^1 \cong S^{R[G/H]} \quad S^{R[G/H]} \wedge S^V \cong S^V \quad \square$$

Let $X \in \text{FinGpd}$, let $Q_X = \{p_{\otimes}(S^1) \mid p: Y \rightarrow X\} \subset (\mathbb{S}_X)_*$.

If $f: Y \rightarrow X$, $f^*(Q_X) \subset Q_Y$ and $f_*(Q_Y) \subset Q_X$.

$$\Rightarrow \begin{aligned} \text{Span}(\text{FinGpd}) &\longrightarrow \text{Cat}_{\infty}(\text{OCat}_{\infty}^{\text{aff}}) \longrightarrow \text{Cat}_{\infty}(\text{CAlg}_{\infty}^{\text{aff}}) \\ X &\longmapsto ((\mathbb{S}_X)_*, Q_X) \xrightarrow{\text{functor}} \text{Sp}_X \end{aligned}$$



$$\text{Map}_{\text{Span}(\text{FinGpd})}(X, Y) = \text{FinGpd}/\overset{\sim}{X \times Y}$$

$$\text{Map}_{\text{Glo}^+}(X, Y) = \text{Fin}_{X \times Y} \underset{\tau_{\leq 0}}{\simeq}$$

$$\begin{array}{c} f \\ \swarrow \quad \downarrow p \\ X \end{array} \quad \begin{array}{c} g \\ \searrow \quad \downarrow q \\ Y \end{array}$$

$\tilde{f} \xrightarrow{p} \tilde{g} \rightarrow X \times Y$ linear/0-trunc.

$$\begin{array}{ccc} \text{Fin}_X & \xrightarrow{f^*} & \text{Fin}_Y \\ \tilde{f}^* \uparrow & \downarrow p^* & \uparrow g^* \\ \text{Fin}_{\tilde{Z}} & \xrightarrow{\tilde{p}^*} & \text{Fin}_Y \end{array}$$

$$\begin{aligned} \text{id} &\xrightarrow{\sim} p \circ p^* \\ \text{wlog } p: BG &\rightarrow * \\ p \circ p^*(A) &= \lim_{BG} A = A \end{aligned}$$

Repeat the above construction:

$$\begin{aligned} \text{Glo}^+ &\longrightarrow \text{Cat}_{\infty}^{\text{soft}} \\ X &\longmapsto \text{Sp}_X \end{aligned}$$

Example:

$$\begin{array}{c} * \xleftarrow{B_G} * \\ \downarrow \text{fin}^G \quad \downarrow \Phi^G \\ \text{fin}^G \quad \Phi^G \end{array}$$

composition in Glo^+ is $* \xleftarrow{+} *$
 $\Rightarrow \Phi^G \text{fin}^G = \text{id}$.