Integral Homotopy Theory Seminar - Week 4

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Outline

Part 0: Recollections on Sp_G.

Recall on Sp_G.

Gpd the 2-category of finite groupoids. Finite means $\pi_0 X$ and $\pi_1 X$ are finite for $X \in \mathrm{Gpd}$. Basically $X \simeq \coprod BG_i$ for some finite collection of finite groups G_i.

Language

Spectra with G-action $= \mathrm{Fun}(BG, \mathrm{Sp})$. a.k.a. "Borel"
G-equivariant spectra $= \mathrm{Sp}_G$. a.k.a. "genuine"

Functoriality of $G \mapsto \mathrm{Sp}_G$

- Contravariant f^*:
  1. Restriction: $H \subset G$ gives
     $$res^G_H : \mathrm{Sp}_G \to \mathrm{Sp}_H \, .$$
  2. "Trivial" action: $G \twoheadrightarrow G/N$ gives
     $$triv^G_{G/N} : \mathrm{Sp}_{G/N} \to \mathrm{Sp}_G \, .$$
- Covariant $g_\otimes$:
  3. Norm: $H \subset G$ gives
     $$N^G_H : \mathrm{Sp}_H \to \mathrm{Sp}_G \, .$$
  4. Geometric fixed points: $G \twoheadrightarrow G/N$ gives
     $$\Phi^N : \mathrm{Sp}_G \to \mathrm{Sp}_{G/N} \, .$$

Three fixed points

Geometric $\Phi^G : \mathrm{Sp}_G \to \mathrm{Sp}$

Preserves all colimits. And

$$\Phi^G(\Sigma^\infty S) = \Sigma^\infty(S^G) \qquad \text{“geometry”}$$

Categorical $(-)^G : \mathrm{Sp}_G \to \mathrm{Sp}$

Right adjoint to $triv^G$, in particular preserves all limits.

Homotopy $(-)^{hG} : \mathrm{Sp}_G \to \mathrm{Sp}$

$X^{hG} = \lim_{BG} X$. Where $X \in \mathrm{Fun}(BG, \mathrm{Sp})$. Only depends on underlying spectrum with G-action.

Spans

Theorem

There is a functor

$$\Psi : \mathrm{Span}(\mathrm{Gpd}) \to \mathrm{Cat}_\infty$$

sending a finite groupoid X to Sp^X. For $X = BG$ have $\mathrm{Sp}^{BG} = \mathrm{Sp}_G$. A span

$$X \xleftarrow{f} M \xrightarrow{g} Y$$

is sent to the functor $g_\otimes f^* : \mathrm{Sp}^X \to \mathrm{Sp}^Y$.

Refinement

Refinement

The category Glo^+: objects are still finite groupoids. Morphisms from X to Y are finite covering maps $M \to X \times Y$. Composition with $N \to Y \times Z$, given by factorization

$$M \times_Y N \to T \to X \times Z$$

where first map has connected fibers and second is finite covering.

Have factorization

$$\mathrm{Span}(\mathrm{Gpd}) \xrightarrow{\pi} \mathrm{Glo}^+ \xrightarrow{\Psi^+} \mathrm{Cat}_\infty$$

Spans - Examples

Restriction

For $H \subset G$ the span

$$BG \xleftarrow{f} BH \xrightarrow{id} BH$$

induces the functor $f^* = res^G_H : \mathrm{Sp}_G \to \mathrm{Sp}_H$.

Geometric fixed points

For $G \to G/N$ the span

$$BG \xleftarrow{id} BG \xrightarrow{g} B(G/N)$$

induces the functor $g_\otimes = \Phi^N : \mathrm{Sp}_G \to \mathrm{Sp}_{G/N}$.

Spans - Examples 2

Composition of spans lead to relations between functors.

The composite

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ BC_p & & * \end{array} \quad \circ \quad \begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ * & & BC_p \end{array} \quad = \quad \begin{array}{ccc} & C_p & \\ \swarrow & & \searrow \\ * & & * \end{array}$$

encodes the relation

$$res_e^{C_p} \circ N_e^{C_p} = (-)^{\otimes p}$$

Spans - Examples 3

The composite in Glo^+

$$\begin{array}{ccc} & BC_p & \\ \swarrow & & \searrow \\ BC_p & & * \end{array} \circ \begin{array}{ccc} & BC_p & \\ \swarrow & & \searrow \\ * & & BC_p \end{array} = \begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ * & & * \end{array}$$

encodes the relation

$$\Phi^{C_p} \circ triv^{C_p} = id$$

Note than in Span(Gpd) this would give the span $* \leftarrow BC_p \rightarrow *$ instead. But in Glo^+ we use the factorization

$$BC_p \rightarrow * \rightarrow * \times * = *$$

of connected fibers followed by finite cover ("finite cover" means potentially non-surjective).

Spans - Examples 4

Let $H \subset G$. The composite

$$\begin{array}{ccc} & BG & \\ \swarrow & & \searrow \\ BG & & * \end{array} \circ \begin{array}{ccc} & BH & \\ \swarrow & & \searrow \\ BH & & BG \end{array} = \begin{array}{ccc} & BH & \\ \swarrow & & \searrow \\ BH & & BG \end{array}$$

encodes the relation

$$\Phi^G \circ N_H^G = \Phi^G$$

In particular

$$\Phi^G \circ N^G = id.$$

Example: Genuine C_p-spectra

Genuine C_p-spectra

Giving $E \in \mathrm{Sp}_{C_p}$ is equivalent to giving a triple (E_0, E_1, f) where:

1. $E_0 \in \mathrm{Fun}(BC_p, \mathrm{Sp})$ "is" the underlying spectrum with C_p-action.
2. $E_1 \in \mathrm{Sp}$ "is" the geometric fixed points $\Phi^{C_p} E$.
3. $f : E_1 \to E_0^{tC_p}$ is a map of spectra.

Recovring fixed points

The categorical fixed points $E^{C_p} \in \mathrm{Sp}$ of $E = (E_0, E_1, f)$ may be recovered as the following pullback

$$\begin{array}{ccc} E^{C_p} & \longrightarrow & E_1 \\ \downarrow & & \downarrow f \\ E_0^{hC_p} & \xrightarrow{\text{can}} & E_0^{tC_p} \end{array}$$

Borel and Borelification

Proposition

Have a forgetful-cofree adjunction

$$U : \mathrm{Sp}_G \rightleftharpoons \mathrm{Fun}(BG, \mathrm{Sp}) : r_G$$

Where the right adjoint is fully faithful.

Definitions

- The essential image $\mathrm{Sp}_G^{Borel} \subset \mathrm{Sp}_G$ of r_G is (by definition) the subcategory of *Borel* G-spectra.
- The composite

  $$\beta = \beta_G : \mathrm{Sp}_G \to \mathrm{Fun}(BG, \mathrm{Sp}) \to \mathrm{Sp}_G$$

  is called the *Borelification*. The unit gives a natural transformation $id \to \beta$.

Borel and Borelification, proof

Proposition

Have a forgetful/cofree adjunction

$$U : \mathrm{Sp}_G \rightleftharpoons \mathrm{Fun}(BG, \mathrm{Sp}) : r_G$$

Where the right adjoint is fully faithful.

Proof sketch

Choose point-set model $\widetilde{\mathrm{Sp}_G}$ for Sp_G and choose free contractible G-space EG. Define

$$\widetilde{\beta} : \widetilde{\mathrm{Sp}_G} \to \widetilde{\mathrm{Sp}_G} \quad X \mapsto \mathrm{Map}(EG, X)$$

Check that this factors through subcategory of Borel G-spectra (i.e. spectra where $(-)^H = (-)^{hH}$ for all subgroups H). Also have natural transformation $id \to \tilde{\beta}$ given by $const : X \to \mathrm{Map}(EG, X)$. Check that this descends to ∞-categories, to give $\beta : \mathrm{Sp}_G \to \mathrm{Sp}_G$ which is "idempotent". Now Lurie [5.2.7.4] tells us this is a localization (with fully faithful right adjoint).

Part 1: Tate diagonal in Sp_G.

Recall: Tate construction and diagonal

- G finite group, and $X \in \mathrm{Fun}(BG, \mathrm{Sp})$. Have norm map

$$Nm : X_{hG} \to X^{hG}$$

  and define X^{tG} as the cofiber. So have cofiber sequence

$$X_{hG} \to X^{hG} \to X^{tG}.$$

- For $G = C_p$, have "Tate diagonal"

$$\Delta_p : X \to (X^{\otimes p})^{tC_p}$$

  Defined using "Yoneda trick". Used that

$$X \mapsto (X^{\otimes p})^{tC_p}$$

  is exact (by binomial formula).
- This is special feature of spectra. There is no such (lax sym. mon.) non-zero map in $D(\mathbb{Z})$.

Geometric fixed points and Tate construction

Proposition

Let $E \in \mathrm{Sp}_{C_p}$. Then $E^{tC_p} \simeq \Phi^{C_p}(\beta E)$.

Proof sketch: Using isotropy separation sequence and Adam's isomorphism one always has a fiber sequence:

$$X_{hC_p} \to X^{C_p} \to \Phi^{C_p} X$$

Now apply this to $X = \beta E$ and compare with the fiber sequence defining E^{tG}. Use that $(\beta E)^{C_p} \simeq (\beta E)^{hC_p}$.

Definition

Let $E \in \mathrm{Sp}_G$. The *proper Tate construction* $E^{\tau G}$ is defined as $\Phi^G(\beta E)$.

$$(-)^{\tau G} : \mathrm{Sp}_G \to \mathrm{Sp}$$

admits lax symmetric monoidal structure, since both Φ^G and β do.

Norm and Tate diagonal

G finite, $E \in \mathrm{Sp}_G$. Have norm $N^G : \mathrm{Sp} \to \mathrm{Sp}_G$ and unit $E \to \beta E$ in Sp_G. Get

$$N^G(-) \to \beta N^G(-).$$

Applying Φ^G, get the composite

$$\Delta^G : X \simeq \Phi^G N^G(X) \to \Phi^G(\beta N^G(X)) \simeq (X^{\otimes G})^{\tau G}$$

This agrees with the Tate diagonal $X \to (X^{\otimes p})^{tC_p}$ when $G = C_p$.

Definition

The above composition defines the *Tate diagonal for G*

$$\Delta^G : (-) \to ((-)^{\otimes G})^{\tau G} : \mathrm{Sp} \to \mathrm{Sp}\,.$$

Part 2: The $\mathbb{E}_\infty$-Frobenius

Frobenius

Fix prime p.
For discrete ring R have two maps

$$\text{can} : R \to R/p \qquad x \mapsto x \pmod{p}$$

and

$$\phi : R \to R/p \qquad x \to x^p \pmod{p}$$

which is also ring homomorphism:

$$(x+y)^p = x^p + y^p \pmod{p}$$

using the binomial formula.

$\mathbb{E}_\infty$-Frobenius

Likewise for $A \in \mathrm{CAlg}$...

Definition

Let $A \in \mathrm{CAlg}$. The $\mathbb{E}_\infty$-Frobenius on A, is the ring-map ϕ_p defined as the composition

$$\phi_p : A \xrightarrow{\Delta_p} (A^{\otimes p})^{tC_p} \xrightarrow{mult_A} A^{tC_p}$$

of Tate diagonal with multiplication in A.

Also have a canonical map

$$\mathrm{can} : A \longrightarrow A^{hC_p} \longrightarrow A^{tC_p}$$

Both maps are rings maps.

Frobenius on Eilenberg-Mac Lane

Discrete rings

Let $A = HR \in \mathrm{CAlg}$ for discrete ring R and $\phi_p : HR \to (HR)^{tC_p}$ be the $\mathbb{E}_\infty$-Frobenius. Then

$$\pi_0(\phi_p) : R \simeq \pi_0(HR) \to \pi_0(HR^{tC_p}) \simeq R/p$$

is the ordinary Frobenius $x \mapsto x^p \pmod{p}$.

Frobenius and Steenrod squares

The $\mathbb{E}_\infty$-Frobenius ϕ does induce the ordinary Frobenius on π_0. But converse is not true: Consider $\mathbb{F}_2 = H\mathbb{F}_2$ with trivial C_2-action. Then

$$\pi_*(\mathbb{F}_2^{tC_2}) \simeq \hat{H}^*(C_2, \mathbb{F}_2) \simeq \mathbb{F}_2((s)) \qquad s \in \pi_1$$

Get spectrum level splitting:

$$\mathbb{F}_2^{tC_2} \simeq \prod_{n\in\mathbb{Z}} \Sigma^n \mathbb{F}_2$$

Theorem (Nikolaus-Scholze)

The $\mathbb{E}_\infty$-Frobenius $\phi : \mathbb{F}_2 \to (\mathbb{F}_2)^{tC_2}$ is the product of all non-negative Steenrod squares $sq^n : \mathbb{F}_2 \to \Sigma^n\mathbb{F}_2$ for $n \geq 0$.

Frobenius and Segal conjecture

Segal's conjecture for C_p may be phrased as the following theorem.

Theorem (Gunawardena, Lin)

For $A = \mathbb{S}$ both maps $\mathrm{can}, \phi_p : \mathbb{S} \to \mathbb{S}^{tC_p}$ exhibit $\mathbb{S}^{tC_p}$ as the p-completion of $\mathbb{S}$.

Remark

Once one has the theorem for can then it follows for ϕ_p, by multiplicativity. Indeed both can and ϕ_p are $\mathbb{S}$-algebra maps, hence $\mathrm{can} = \phi_p$.

Frobenius and Adams op's

Consider $A = \mathrm{KU}$ the periodic complex K-theory spectrum, equipped with trivial C_2-action. Recall that $\pi_* KU \simeq \mathbb{Z}[\beta^{\pm 1}]$. Can show

$$\pi_*(\mathrm{KU}^{tC_p}) \simeq \pi_*(\mathrm{KU})((t))/((t+1)^p - 1) \simeq \pi_* \mathrm{KU} \otimes \mathbb{Q}_p(\zeta_p).$$

Theorem (Nikolaus-Scholze)

Suppose X a retract of finite CW-complex. Then Frobenius $\phi_p : \mathrm{KU} \to \mathrm{KU}^{tC_p}$ induces the map

$$\mathrm{KU}^0(X) \to \mathrm{KU}^0(X) \otimes \mathbb{Q}_p \qquad V \mapsto \psi^p(V)$$

where $\psi^p = p$-th Adams operation.

Complements on Frobenius and Power operations

Nikolaus and Scholze use similar strategy to identify both $\phi_2 : \mathbb{F}_2 \to (\mathbb{F}_2)^{tC_2}$ and $\phi_p : \mathrm{KU} \to \mathrm{KU}^{tC_p}$. Let $A \in \mathrm{CAlg}$. Their strategy is to reduce to identifying action of ϕ_p on associated cohomology theory $A^*(X) = [\Sigma^{-*}X, A]$ in terms of *power operations*. Power operations are stable operations on multiplicative cohomology which are constructed from the *space level* diagonal.

Proposition

For $X = \Sigma^\infty_+ Y$ there is a unique (lax sym. mon.) factorization

$$\begin{array}{ccc} & & (\Sigma^\infty_+(Y \times \cdots \times Y))^{h\Sigma_p} \\ & \overset{\Sigma^\infty_+\Delta}{\nearrow} & \downarrow \\ X & \xrightarrow{\Delta_p} & (X \otimes \cdots \otimes X)^{tC_p} \end{array} \tag{1}$$

Frobenius and Power operations, 2

Proposition

For $X = \Sigma^\infty_+ Y$ there is a unique (lax sym. mon.) factorization

$$\begin{array}{ccc} & & (\Sigma^\infty_+(Y \times \cdots \times Y))^{h\Sigma_p} \\ & \overset{\Sigma^\infty_+\Delta}{\nearrow} & \downarrow \\ X & \xrightarrow{\Delta_p} & (X \otimes \cdots \otimes X)^{tC_p} \end{array} \tag{2}$$

Proof: By a theorem of Nikolaus, $\Sigma^\infty_+ : \mathrm{Spc} \to \mathrm{Sp}$ is initial lax sym. mon. functor between the two categories. Since all three functors are lax. sym. mon. it follows that the diagram commutes.

Frobenius and Power operations, 3

Corollary

For an $\mathbb{E}_\infty$-ring A, the map

$$\Omega^\infty \phi_p : \Omega^\infty A \to (\Omega^\infty A^{tC_p})^{h\mathbb{F}_p^\times}$$

factors as

$$\Omega^\infty A \xrightarrow{\Delta} ((\Omega^\infty A)^{\times p})^{h\Sigma_p} \xrightarrow{mult} (\Omega^\infty A)^{h\Sigma_p} \xrightarrow{can} (\Omega^\infty A^{tC_p})^{h\mathbb{F}_p^\times}$$

Here the composite

$$P_p : \Omega^\infty A \xrightarrow{\Delta} ((\Omega^\infty A)^{\times p})^{h\Sigma_p} \xrightarrow{mult} (\Omega^\infty A)^{h\Sigma_p}$$

is the power operation.
(For more details, see Nikolaus-Scholze section IV.1)

Part 3: The generalized $\mathbb{E}_\infty$-Frobenius

Generalized Frobenius - the idea

Let $A \in \mathrm{CAlg}$ be an $\mathbb{E}_\infty$-algebra, and G a finite group.
The $\mathbb{E}_\infty$-multiplication on A gives a map

$$A^{\otimes G} \to A$$

of spectra with G-action. Where $A^{\otimes G}$ is the underlying spectrum of $N^G A$. Since βA is cofree on A (as a spectrum with trivial G-action) we get an induced map

$$N^G A \to \beta A.$$

Applying Φ^G gives a map

$$\phi^G : A \simeq \Phi^G N^G A \to \Phi^G \beta A \simeq A^{\tau G}$$

Which is called the *generalized Frobenius map*.

Generalized Frobenius - more detail

The underlying spectrum of $N^G A$ is just $A^{\otimes G}$ (the G-indexed tensor product). The $\mathbb{E}_\infty$-multiplication on A then gives a map

$$A^{\otimes G} \to A$$

of spectra with G-action. I.e. it is a map

$$U_G N^G A \to U_G triv^G A$$

Now, $\beta triv^G A = r_G(U_G triv^G A)$ is the cofree genuine G-spectrum on $U_G triv^G A$. So the above map (from the underlying spectrum of a genuine G-spectrum) induces a map

$$N^G A \to \beta triv^G A.$$

Applying Φ^G gives a map

$$\phi^G : A \simeq \Phi^G N^G A \to \Phi^G \beta triv^G A \simeq A^{\tau G}$$

Which is called the *generalized Frobenius map*.

Even more generalized Frobenius

As before, but now given $H \subset G$ a subgroup. The $\mathbb{E}_\infty$-multiplication on A gives a map

$$A^{\otimes G/H} \to A$$

of spectra with G-action. Where $A^{\otimes G/H}$ is the underlying spectrum of $N_H^G \beta_H triv^H A$. Since $\beta_G A$ is cofree on A (as a spectrum with trivial G-action) we get an induced map

$$N_H^G \beta_H triv^H A \to \beta_G A.$$

Applying Φ^G gives a map

$$\phi_H^G : A^{\tau H} \simeq \Phi^H \beta_H triv^H A \simeq \Phi^G N_H^G \beta_H triv^H A \to \Phi^G \beta_G A \simeq A^{\tau G}$$

Which is called the *generalized Frobenius map*.

Generalized canonical map

Let A be any spectrum (e.g an $\mathbb{E}_\infty$-ring). Let $G \twoheadrightarrow K$ surjection of finite groups. Have

$$triv_K^G \beta_K triv^K A \to \beta_G triv^G A$$

applying Φ^G get

$$A^{\tau K} \simeq \Phi^K \beta_K triv^K A \simeq \Phi^G triv_K^G \beta_K triv^K A \to \Phi^G \beta_G triv^G A \simeq A^{\tau G}$$

i.e. a map

$$\mathrm{can}_K^G : A^{\tau K} \to A^{\tau G}$$

called the generalized canonical map.

Remark

When A is an $\mathbb{E}_\infty$-ring, can_K^G is multiplicative.

Special case

When $H = e \subset G$ is trivial subgroup, $\phi_H^G = \phi^G$ is the composite of Tate diagonal with multiplication

$$\phi^G : A \to (A^{\otimes G})^{\tau G} \to A^{\tau G}$$

In particular for $G = C_p$ recover the Nikolaus-Scholze Frobenius.

Part 4: The Frobenius action on CAlg

The category $\mathcal{Q}$

FinGrp $\stackrel{def}{=}$ category of finite groups. Then,

$$\mathcal{Q} \stackrel{def}{=} \text{Quillen's Q-construction on FinGrp}$$

except, FinGrp is not an exact category...

Official definition

The symmetric monoidal 1-category $\mathcal{Q}$ has

- Objects: finite groups
- Morphism: from H to G: (surjection,injection)-spans $H \twoheadleftarrow K \hookrightarrow G$
- Composition: usual composition of spans
- Symmetric monoidal structue: Cartesian product

Integral Forbenius action

Theorem $A^\sharp$ (Yuan)

There is an oplax monoidal functor

$$\mathcal{Q} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{CAlg})$$

such that

1. $G \in \mathcal{Q}$ acts by the proper Tate construction $(-)^{\tau G}$.
2. A surjection $(K \twoheadleftarrow G = G)$ is sent to the canonical transformation

$$\mathrm{can}_K^G : (-)^{\tau K} \longrightarrow (-)^{\tau G}.$$

3. An injection $(H = H \hookrightarrow G)$ is sent to the Frobenius

$$\phi_H^G : (-)^{\tau H} \longrightarrow (-)^{\tau G}.$$

Oplax monoidal?

Giving a $F : C \to D$ between monoidal an oplax structure means giving suitably compatible maps

$$F(c \otimes_C c') \to F(c) \otimes_D F(c').$$

In particular the oplax'ness of

$$\mathcal{Q} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{CAlg})$$

means we have suitably compatible transformations

$$(-)^{\tau(G \times H)} \longrightarrow ((-)^{\tau H})^{\tau G}$$

for all finite groups G, H.
See Remark 3.14 for explicit construction.

Baby application

Suppose $A \in \mathrm{CAlg}$ is p-complete and that

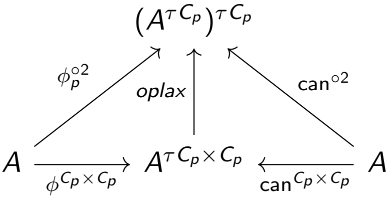
$$\mathrm{can}^{C_p} : A \to A^{\tau C_p} \tag{3}$$
$$\mathrm{can}^{C_p \times C_p} : A \to A^{\tau C_p \times C_p} \tag{4}$$
$$\phi_p : A \to A^{\tau C_p} \tag{5}$$

are equivalences. Then may conclude that also

$$\phi^{C_p \times C_p} : A \to A^{\tau C_p \times C_p}$$

is an equivalence.

Proof: Have a commutative diagram



Easy to show that iterated ϕ_p and *can* are equivalences. Thus all outer arrows, except $\phi^{C_p \times C_p}$, are equivalences, hence it is too.