

Partial K-theory: S.-construction

Goal for today: non-group complete analogue of K-theory, via an analogue of S.-construction.

[Next time: partial K-theory via analogue of Q-construction and then that they agree.
 $K^{\text{part}}(\mathbb{F}_p)$ will play an important role further...]

Main idea: \mathcal{C} -Waldhausen ∞ -category vs
 define an E_∞ -space $K^{\text{part}}(\mathcal{C})$ s.t.:

- 1) $K^{\text{part}}(\mathcal{C})^{\text{gp}} \simeq K(\mathcal{C})$ can. equiv. of E_∞ -spaces
- 2) $\mathbb{S}_0 K^{\text{part}}(\mathcal{C})$ is the free monoid on \mathcal{C} with relations $[B] = [A] + [C]$ for s.e.s. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Recollection on S.-construction

- Waldhausen ∞ -category is $(\mathcal{C}, \mathcal{C}_+)$ where $\mathcal{C}_+ \subset \mathcal{C}$ is a subcategory of "cofibrations" containing \mathcal{C}^\sim s.t.
 - 1) zero object exists and $x \rightarrow x$ are cofibrations $\forall x \in \mathcal{C}$
 - 2) pushouts of cofibrations exist and are cofibrations

Examples: 1) \mathcal{C} pointed ∞ -cat. with finite colims (e.g. stable).

Then $\mathcal{C}_+ = \mathcal{C}$ makes sense to take

2) $\mathcal{C} = \text{fin. dim. } \mathbb{F}_p\text{-v.sp.}$, \mathcal{C}_+ = subcat. of injective maps.

- S.-construction of $(\mathcal{C}, \mathcal{C}_+)$ (Barwick) is a simplicial ∞ -cat. $S(\mathcal{C})$:
- $$S_n(\mathcal{C}) = \left\{ X(0,0) \hookrightarrow X(0,1) \hookrightarrow \dots \hookrightarrow X(0,n) \right. \quad \left. \begin{array}{l} X(i,i) \simeq * \forall i \\ \text{s.t.} \end{array} \right\}$$
- $$\left\{ \begin{array}{c} \downarrow \\ X(1,1) \hookrightarrow \dots \hookrightarrow X(0,n) \\ \vdots \\ \dots \dots X(h,h) \end{array} \right. \quad \left. \begin{array}{l} \text{all squares are} \\ \text{pushouts} \end{array} \right\}$$

In particular, $S_0(\mathcal{C})$ is contractible; $S_1(\mathcal{C}) \simeq \mathcal{C}$;
 $S_n(\mathcal{C}) \simeq \{x \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_n \mid X_i \in \text{Ob}(\mathcal{C})\}$ - sequences of cofibrations.

We need gapped objs to describe simplicial structure on $S_2(\mathcal{C})$, for example, face maps on $S_2(\mathcal{C})$ are given by:

$$d_0(X) = \left\{ \begin{matrix} X(1,1) \hookrightarrow X(1,2) \\ \downarrow \\ X(2,2) \end{matrix} \right\}, \quad d_1(X) = \left\{ \begin{matrix} X(0,0) \hookrightarrow X(0,2) \\ \downarrow \\ X(1,2) \end{matrix} \right\};$$

we need to specify cofiber, if its existence upto contractible choice

$$d_2(X) = \left\{ \begin{matrix} X(0,0) \hookrightarrow X(0,1) \\ \downarrow \\ X(1,1) \end{matrix} \right\}$$

- Waldhausen K-thy is the E_1 -space (in fact, E_∞)
 $K(\mathcal{C}) = \bigvee S_*(\mathcal{C})$ take max. subgroupoid levelwise,
and $H: \text{Fun}(\Delta^{\text{op}}, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}$: const. So $K_n(\mathcal{C}) \simeq \pi_{n+1}(S_*(\mathcal{C})^\sim)$.

Morally: $K(\mathcal{C})$ can be thought of as the universal grouplike E_1 -space obtained out of $S_*(\mathcal{C})^\sim$, and we want a non-grouplike version.

To define it, we need to recall:

A Segal space is $X(-): \Delta^{\text{op}} \rightarrow \mathcal{S}$ s.t. $\forall n \geq 1$
Segal maps $j_i: [1] \rightarrow [n]$ $0 \mapsto i$, $1 \mapsto i+1$ induce
 $\prod_{i=0}^{n-1} X(j_i): X([n]) \xrightarrow{\sim} X([1]) \times \dots \times X([n])$ category object in \mathcal{S}

There's a fully faithful embedding using Ass
 $(B: \text{Mon}(\mathcal{S}) \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathcal{S}))$ $M \mapsto (\text{id} \in M \subseteq M \times M \dots)$ bare construction

whose ess. image is the subcat. of Segal spaces
X s.t. $X([0]) \cong *$.

Since the full subcat. $\text{Mon}(S)$ is closed under
lims and filtered colims, there's a left adjoint

$$\mathbb{L}: \text{Fun}(\Delta^{\text{op}}, S) \rightleftarrows \text{Mon}(S): \mathbb{R},$$

def. \mathcal{C} Waldhausen ∞ -cat. Then

$$\underline{K^{\text{part}}(\mathcal{C})} := \mathbb{L}(S_*(\mathcal{C})^{\simeq})$$

Remark: the moral can be made precise as follows.

We will prove that $K(\mathcal{C}) \cong K^{\text{part}}(\mathcal{C})^{\text{gp}}$,
hence $K(\mathcal{C}) \cong (\mathbb{L}(S_*(\mathcal{C})^{\simeq}))^{\text{gp}} \cong \mathbb{G}(S_*(\mathcal{C})^{\simeq})$, where

$$\mathbb{G}: \text{Fun}(\Delta^{\text{op}}, S) \rightleftarrows \text{Grp}(S): \mathbb{R},$$

i: $\text{Grp}(S) \hookrightarrow \text{Mon}(S)$ is the full subcat. of grouplike \mathbb{G} -spaces.

\cong holds because right adjts commute \Rightarrow so do

left adjoints; I don't know when $\mathbb{L}|_{\mathcal{C}} \cong \mathbb{G}$ in general.
(true for spaces X s.t. $X_0 \cong *$).

We want to rewrite K^{part} in terms of CSS.

Recall: $\text{CplSeg}(S) \subset \text{Seg}(S)$ are complete Segal spaces;
there's an adjunction $\text{CSS}: \text{Fun}(\Delta^{\text{op}}, S) \rightleftarrows \text{CplSeg}(S)$,
and most importantly,

$$\text{Cat}_{\infty} \cong \text{CplSeg}(S)$$

$$\mathcal{C} \mapsto \text{Fun}(\Delta^*, \mathcal{C})^{\simeq}$$

Under this identification, CSS is the colimit-preserving
functor that sends $[n]$ to $[n]$ regarded as ∞ -cat.

Complete: there's an internal definition, a Segal space X_0 is complete iff $X_0 \cong \text{Map}(*, X_0) \xrightarrow{\text{So}} \text{Map}(N(\cdot \tilde{\rightarrow} \cdot), X_0) \hookleftarrow_{\text{equivalence}}$, RHS is a subspace of X_1 of & ex, s.t. $\exists g \in X_1 : fg \sim id$, Non-examples: 1) In $\text{Corr}_1^F(e)$ you can label $\begin{matrix} id_C \\ C \end{matrix} \downarrow \begin{matrix} id_D \\ C \end{matrix}$ by non-trivial 2) BM.

One can extend the picture: invertible items in $F(id_C)$,

$$\text{Cat}_{\infty} \cong \text{CplSeg}(\mathcal{S}) \quad \text{whereas } \text{Corr}_0^F(e) \cong \mathcal{C}^\approx.$$

$$\begin{matrix} \text{forget} \uparrow ? & \uparrow ? \\ \{X \xrightarrow{f} e\} = \text{Cat}_{\infty}^{\text{fl}} & \simeq \text{Seg}(\mathcal{S}) \\ \text{ess. surj. } (Y_0 \xrightarrow{\text{CSS}(Y)} \hookrightarrow Y & \text{co-gpd} \end{matrix}$$

Under this identification, $\text{CplSeg}(\mathcal{S}) \hookrightarrow \text{Seg}(\mathcal{S})$ corresponds to the subcat. of $\text{Cat}_{\infty}^{\text{fl}}$ consisting of $\{X \xrightarrow{f} e \mid f \simeq: X^\approx \rightarrow e^\approx\}$.

In this language, we obtain:

$$\text{Mon}(\mathcal{S}) \cong (\text{Cat}_{\infty})_0 = \{ * \xrightarrow{f} e \} \quad \text{space!}$$

$$M \mapsto \text{BM}: \text{Ob} = *; \text{Hom}(*, *) = M.$$

[you can do this for any \mathcal{E}_n , not just \mathcal{E}_1 :
 $\text{Mon}_{\mathcal{E}_n}(\mathcal{S}) \cong \{ e \rightarrow e; e \text{ } \mathcal{E}_{n-1}\text{-monoidal } \infty\text{-cat}\}$.]

Main prop: e -Waldhausen ∞ -cat. Then:

$$1) \text{Hom}_{\text{CSS}(S.(e)^\approx)}(*, *) \cong K^{\text{part}}(e) \text{ as } \mathcal{E}_1\text{-spaces}$$

$$2) \text{CSS}(S.(e)^\approx) \cong BK^{\text{part}}(e) \text{ as } \infty\text{-cats.}$$

(prove latter)

Rephrase with a slogan:

K-theory is to spaces what partial K-theory is to n -categories.

Because: $BK(\mathcal{C}) = B\mathcal{J}_\mathcal{C} | S_+(\mathcal{C})^\simeq | \simeq | S_+(\mathcal{C})^\simeq |$

because $|S_+(\mathcal{C})^\simeq|$ is connected,
and $B\mathcal{J}_\mathcal{C}$ = pick the comp. of base pt

And we have: $\text{Fun}(\Delta^{\text{op}}, S) \xrightarrow[\text{const}]{} S \ni BK(\mathcal{C})$

important diagram

$$\begin{array}{ccc} S_+(\mathcal{C})^\simeq & \xrightarrow{H} & S \\ \text{CSS} \swarrow & \uparrow & \searrow \text{invertible} \\ \text{CSS} & \xrightarrow{\quad i \quad} & \text{CAlg}(S)^\simeq = \text{Cat}_\infty \ni BK^{\text{part}}(\mathcal{C}) \end{array}$$

Cor. 1. Coproduct on \mathcal{C} makes $K^{\text{part}}(\mathcal{C})$ an \mathbb{E}_∞ -space.

Proof: • coproduct on \mathcal{C} makes $S_+(\mathcal{C})^\simeq$ an \mathbb{E}_∞ -monoid
in simplicial spaces, because $V: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is an
exact functor (preserves 0, cofibrations and
pushouts along cofibrations)

• $\text{CSS}(S_+(\mathcal{C})^\simeq)$ is a symm. mon. ∞ -cat. because

$\text{CSS}: \text{Fun}(\Delta^{\text{op}}, S) \xrightarrow[\text{const}]{} \text{Cat}_\infty$: i commutes with finite products:
since CSS preserves colims, enough to check on
representables, but they are in the image of i
which does preserve products.
• conclude by Main Prop.

Cor. 2. $K^{\text{part}}(\mathcal{C})^{\text{gp}} \simeq K(\mathcal{C})$ as E_∞ -spaces.

Proof: $K(\mathcal{C}) = \Omega |S.(\mathcal{C})^\simeq| \simeq \Omega_{\text{!`}} |\text{CSS}(S.(\mathcal{C})^\simeq)|$

$$K^{\text{part}}(\mathcal{C})^{\text{gp}} \stackrel{\text{Main Prop}}{\simeq} \text{Hom}_{\text{CSS}(S.(\mathcal{C})^\simeq)}^{(\ast, \ast)^{\text{gp}}} \xrightarrow{\text{!`}} \text{Hom}_{\text{CSS}(S.(\mathcal{C})^\simeq)}^{(\ast, \ast)} \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

from Important Diagram, $|\text{CSS}(S.(\mathcal{C})^\simeq)|$ is equivalent to $\text{CSS}(S.(\mathcal{C})^\simeq)$ with all maps inverted, and since it has just one object by Main Prop(2), on the level of $\text{End}(\ast)$ it's literally group completion. It's an equivalence of E_∞ -spaces, because

functors under consideration commute with finite prods.

Or: $K^{\text{part}}(\mathcal{C}) \rightarrow K(\mathcal{C})$ is a map of E_∞ -spaces.

Prop. $K_0^{\text{part}}(\mathcal{C}) := \pi_0 K^{\text{part}}(\mathcal{C})$ is the monoid freely generated by $\text{Ob}\mathcal{C}$ modulo the relation $[B] = [A] + [C]$ for s.c.s. $A \xrightarrow{i} B \rightarrow C \cong \text{coker}(i)$

Proof. Moral: >2 -simplices of $S.$ don't contribute to relations:
Consider comm. diagram

$$\text{Fun}(\Delta^{\text{op}}, S) \xrightleftharpoons[\text{!`}]{} \text{Mon}(S)$$

$$\text{Fun}(\Delta^{\text{op}}, \text{Set}) \xrightleftharpoons[\text{!`}]{} \text{Mon}(\text{Set})$$

$$i: \Delta^{\text{op}} \hookrightarrow \Delta^{\text{op}}$$

≤ 2
full subcat.

on $[\mathcal{C}_0], [\mathcal{C}_1], [\mathcal{C}_2]$

$$\text{Fun}(\Delta_{\leq 2}^{\text{op}}, \text{Set}) \xrightleftharpoons[\text{!`}]{} \text{Mon}(\text{Set})$$

$$i^* \downarrow \uparrow i_* = \text{RKE}$$

$$\xrightarrow{I_0} \xleftarrow{B_0}$$

I_0 is just a notation

We get for $X \in \text{Fun}(\Delta^{\text{op}}, S)$

$$\pi_0 \mathbb{L}X \simeq \overline{\mathbb{L}_0} i^* \pi_0 X.$$

Want: compute it for $X = S.(C) \simeq$.

Take $M_0 \in \text{Mon}(\text{Set})$. Then a map $i^* \pi_0(S.C) \xrightarrow{\cong} \overline{\mathbb{B}_0}(M_0)$ is given just by

$$* \mapsto *$$

$$[y] \xrightarrow{f} [f(y)] \in M,$$

and face maps should be preserved \Rightarrow

$$\text{for } x \rightarrow y \text{ get } f([y]) = f([x]) + f([y/x]).$$

Hence $K_0^{\text{part}}(C)$ satisfies the universal property we want.

Warning. C stable ∞ -category \Rightarrow

$$K_0^{\text{part}}(C) \simeq K(C), \text{ because}$$

$$\forall x \quad x \rightarrow 0 \rightarrow \Sigma x \text{ gives } [\Sigma x] = -[x],$$

so $K_0^{\text{part}}(C)$ is already group complete.

Proof of Main Prop. (pretty formal, uses that $\text{So}(C) \simeq *$).
 notation: $f: D \hookrightarrow D'$ we denote f^L and f^R its adjoints if they exist.

$$S.(e) \stackrel{\sim}{\in} \text{Fun}_*(\Delta^{op}, S) \hookrightarrow \text{Fun}(\Delta^{op}, S)$$

simplicial spaces with \xrightarrow{k} factors through k ,
 $X_0 \simeq *$

$$\text{hence } k^{\text{part}}(e) = \mathbb{L} S.(e) \stackrel{\sim}{=} k^L S.(e) \stackrel{\sim}{=}$$

Since $*$ $\in \text{Fun}_*(\Delta^{op}, S)$ is an initial object,
 there's a f.f. embedding

$$i_0: \text{Fun}_b(\Delta^{op}, S) \hookrightarrow \text{Fun}(\Delta^{op}, S)_{*/},$$

which extends to a comm. diagr. of right adj's:

$$\begin{array}{ccccc} \text{CplSeg}_{b/}(S) & \xrightarrow{j_1} & \text{Seg}(S)_{b/} & \xrightarrow{j_0} & \text{Fun}(\Delta^{op}, S)_{b/} \\ \text{categories of} & & \downarrow i_1 & & \downarrow j_0 \\ \text{pointed objects} & & \text{Mon}(S) & \xleftarrow{k} & \text{Fun}_*(\Delta^{op}, S) \end{array}$$

Observe: $\exists i_0^R$, it extracts simplices that involve only the distinguished 0 simplex,
 i.e. $i_0^R(T_0)_n = T_n \times_{(T_0)^{n+1}} *$.

Clearly, j_0^R takes $\text{Seg}(S)_{b/}$ to $\text{Mon}(S)$,
 hence i_1^R exists and

$$(*) \quad i_0^R j_0 \simeq k i_1^R.$$

Now, we're ready to prove 4.8:

$$\begin{aligned} 1) \quad k^{\text{part}}(e) &\simeq k^L(S.(e)) \simeq i_1^R i_1 k^L(S.(e)) \simeq i_1^R j_0^L i_0(S.(e)) \\ &\simeq \underbrace{i_1^R j_0^L j_1^L}_{\cong} j_0^L i_0(S.(e)) \end{aligned}$$

this takes a
pointed ∞ -cat. \mathcal{S}
to $\text{End}(\ast)$.
this is $\text{CSS}(\mathcal{S}(\mathcal{C})^{\approx})$ with
the canonical base point

2) left to show: $\text{CSS}(\mathcal{S}(\mathcal{C})^{\approx}) \simeq \mathcal{B}\text{Mon}(\ast, \ast)$,
i.e. $\ast \rightarrowtail \text{CSS}(\mathcal{S}(\mathcal{C})^{\approx})$.

Equivalence (b) implies that left adj's
commute too, i.e. $i_! k^\perp \simeq j_! i_\circ$. Hence
Segalification of $\mathcal{S}(\mathcal{C})^{\approx}$ factors through
 $\text{Mon}(\mathcal{S})$. As we discussed before,
the resulting $\text{CSS}(\mathcal{S}(\mathcal{C})^{\approx}) \in (\text{Cat}_{\infty})_0$, i.e.
receives an ess. surjection from a point.

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