Partial Algebraic K-theory: Q-construction

1 - Exact ∞-categories

C ∞-category equipped with subcategories

$C_\dagger$ of ingressive maps $\rightarrowtail$

$C^\dagger$ of egressive maps $\twoheadrightarrow$

Def. An ambigressive pullback in C is a pullback

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, two heads] \\ Y' \arrow[r, tail] & Y \end{tikzcd}$$

An ambigressive pushout in C is pushout

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, two heads] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Def. An exact ∞-category C is an ∞-category with $C_\dagger, C^\dagger$ such that:

1) $(C, C_\dagger)$ is a Waldhausen ∞-category

2) $(C^{op}, C^{\dagger op})$ " " "

3) C is additive (hC is additive)

4) Any ambigressive pullback (pushout) is an ambigressive pushout (pullback)

$\rightsquigarrow$ notion of short exact sequence in C

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] \\ 0 \arrow[r, tail] & X'' \end{tikzcd}$$

pushout / pullback

Example:

1) C 1-category

C exact ∞-category $\Leftrightarrow$ exact category in the sense of Quillen

2) $C_+ = C^+ = C$

C exact ∞-category $\Leftrightarrow$ C stable

$$S_n(C) = \left\{ \begin{array}{ccccc} 0 = X_{00} \rightarrowtail & X_{01} \rightarrowtail & \cdots & \rightarrowtail & X_{0n} \\ & \downarrow & & & \downarrow \\ & 0 = X_{11} \rightarrowtail & \cdots & \rightarrowtail & X_{1n} \\ & & \ddots & & \downarrow \\ & & & \ddots & \vdots \\ & & & & \downarrow \\ & & & & X_{nn} = 0 \end{array} \right\}$$

$$S_n(C^{op})^{op} \simeq S_n(C)$$

through symmetry with respect to diagonal

In principle the Q-construction of an exact ∞-category C is the ∞-category of cospans in C of the form

$$Q(C) \qquad \begin{tikzcd} X \arrow[dr, hook] & & Y \arrow[dl, two heads] \\ & Z & \end{tikzcd}$$

$$\mathrm{ob}\, Q(C) \simeq \mathrm{ob}(C)$$

2 - The twisted arrow category

X simplicial set (∞-category)

$$(X^{op})_n = \mathrm{Hom}(\Delta^{n\,op}, X)$$

$$\tilde{\mathcal{O}}(X)_n = \mathrm{Hom}(\Delta^{n\,op} * \Delta^n, X)$$

$$= X_{2n+1}$$

$$\mathcal{O}(X) = \mathcal{O}(X^{op})^{op}$$

$$\Delta^{n\,op} \hookrightarrow \Delta^{n\,op} * \Delta^n \hookleftarrow \Delta^n$$

$$\begin{tikzcd} & \tilde{\mathcal{O}}(X) \arrow[dl] \arrow[dr] & \\ X^{op} & & X \end{tikzcd}$$

Fact: X ∞-category $\Rightarrow$ $\tilde{O}(X) \longrightarrow X^{op} \times X$ is a left fibration

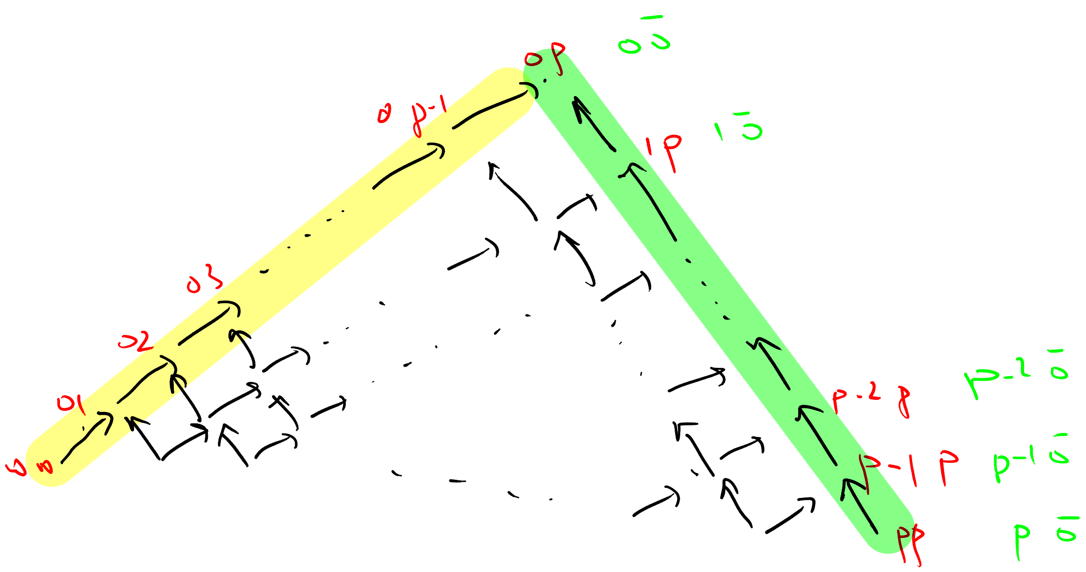
$\Rightarrow$ it classifies a functor

$$\mathrm{Map}_X = \mathrm{Hom}_X : \underset{\left(X \times X^{op}\right)^{op}}{X^{op} \times X} \longrightarrow S$$

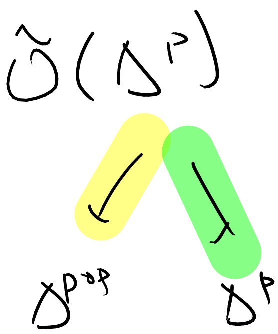
X 1-category $\Rightarrow$ $\tilde{O}(X)$ 1-category.

We will look at $\tilde{O}(\Delta^p)$ (which is a poset).

$$\Delta^{n\,op} \times \Delta^n \longrightarrow \Delta^p$$



$\bar{n} = p - n$



For each $0 \leq i \leq k \leq l \leq j \leq p$

$$\Delta^1 \times \Delta^1 \longrightarrow \tilde{O}(\Delta^p)$$

$$\begin{tikzcd} ij \arrow[r] \arrow[d] & kj \arrow[d] \\ il \arrow[r] & kl \end{tikzcd}$$

3 - Q - construction

C exact ∞-category

Def: $X : \tilde{O}(\Delta^n) \to C$ is ambigressive if for all $0 \le i \le k \le l \le j \le n$

get

$$\begin{tikzcd} X_{ij} \arrow[r, tail] \arrow[d, two heads] & X_{kj} \arrow[d, two heads] \\ X_{il} \arrow[r, tail] & X_{kl} \end{tikzcd}$$

ambigressive pullback

$Q_n(C) \underset{\text{full}}{\subset} \mathrm{Fun}(\tilde{O}(\Delta^n), C)^{\simeq}$

$\shortparallel$

(ambigressive)

$Q_\cdot(C) \subset \mathrm{Fun}(\tilde{O}(\Delta^\cdot), C)^{\simeq}$

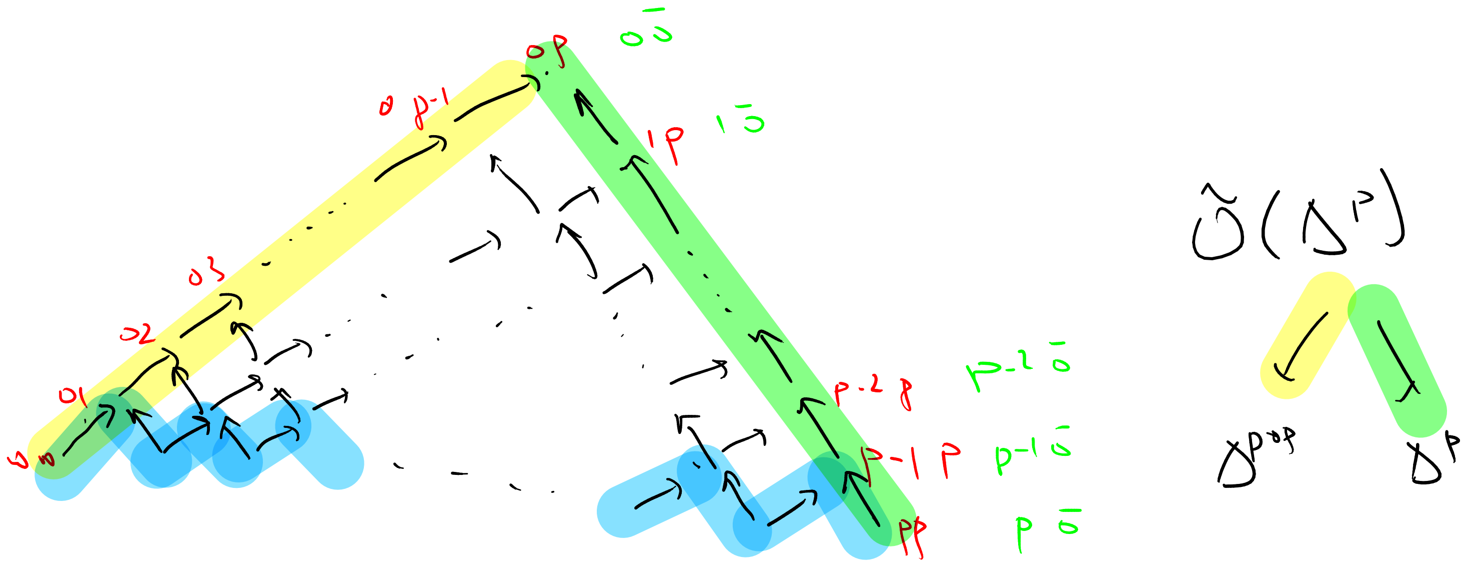
$Q_\cdot(C) \in \mathrm{Fun}(\Delta^{op}, S)$

Def. The Q-construction of C is $Q_\cdot(C)$

Proposition $Q_\cdot(C)$ is a complete Segal space

Segal map

$$\varphi_p(C) \xrightarrow{\simeq} \varphi_1(C) \underset{\varphi_0(C)}{\times} \varphi_1(C) \underset{\varphi_0(C)}{\times} \cdots \underset{\varphi_0(C)}{\times} \varphi_1(C)$$



by uniqueness of pushouts

$\varphi_\bullet(C)$ complete Segal $\quad J \to \Delta^0$ induces

$J = N(\{0 \rightleftarrows 1\})^{\sim}$ nerve

$$C^{\simeq} \cong \varphi_0(C) = \mathrm{Fun}(\Delta^0, C)^{\simeq} \longrightarrow \mathrm{Map}(J, \varphi(C))$$

Lemma: A quasi-category.

$A \in \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \longrightarrow \mathrm{Fun}(\Delta^{op}, S)$

through $\mathrm{Set} \to S$

Define $\varphi_A(C) \underset{\text{full}}{\subset} \mathrm{Fun}(\tilde{O}(A), C)^{\simeq}$

$= \{X : \tilde{O}(A) \to C \mid \forall\, \Delta^n \to A$ the induced functor $\tilde{O}(\Delta^n) \to \tilde{O}(A) \to C$ is ambigressive $\}$

Thm: $\mathrm{Map}(A, \varphi_\bullet(C)) = \varphi_A(C)$

proof: $A = \operatorname{colim}_{\Delta^n \to A} \Delta^n$

$$\begin{tikzcd}
\mathrm{Map}(A, \varphi_\bullet(C)) \arrow[d, dotted] \arrow[r, phantom, "="] & \lim_{\Delta^n \to A} \mathrm{Map}(\Delta^n, \varphi_\bullet(C)) \\
& = \lim_{\Delta^n \to A} \varphi_n(C) \\
\varphi_A(C) \arrow[d, phantom, "\cap"] & \cap \\
\mathrm{Fun}(\tilde{\mathcal{J}}(A), C) \arrow[r, phantom, "\cong"] & \lim_{\Delta^n \to A} \mathrm{Fun}(\hat{\mathcal{J}}(\Delta^n), C)
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{C} = \varphi_0(C) \arrow[r, "\cong"] \arrow[dr, "\cong"'] & \mathrm{Map}(J, \varphi_\bullet(C)) \arrow[d, phantom, "\cong"] \\
& \varphi_J(C) \\
& \mathrm{Fun}(\tilde{\mathcal{J}}(J), C)^{\simeq}
\end{tikzcd}$$

$\varphi_J(C) \cap \; || \; \mathrm{Fun}(\tilde{\mathcal{J}}(J), C)^{\simeq}$

4- Comparison with partial K-theory

$n \geq 0$

$$\tilde{O}(\Delta^n)^{op} = O(\Delta^n) \xrightarrow{\tau_n} O(\Delta^{n\,op} * \Delta^n)$$

$$(i < j) \longmapsto j * i$$

$\Delta^1 \hookrightarrow \Delta^n$

$\Delta^0 \xrightarrow{i} \Delta^n$

$\Delta^0 \xrightarrow{j} \Delta^{n\,op}$

$$\Delta^1 = \Delta^0 * \Delta^0 \xrightarrow{j * i} \Delta^{n\,op} * \Delta^n = \Delta^{2n+1}$$

This map τ is functorial in Δ^n

$$\Delta^{op} \xrightarrow{\varepsilon^{op}} \Delta^{op} \xrightarrow{S(C)} S$$

$$\Delta^n \longmapsto (\Delta^{n\,op} * \Delta^n)^{op}$$

$$\begin{tikzcd}
\varepsilon^* S_\bullet(C)^\simeq \arrow[r, "\tau^*"] \arrow[d, phantom, "\cap"] & Q_\bullet(C) \arrow[d, phantom, "\cap"] \\
\mathrm{Fun}(O(\Delta^{n\,op} * \Delta^n), C) \arrow[r, "\tau_n^*"] & \mathrm{Fun}(\tilde{O}(\Delta^n), C)^\simeq
\end{tikzcd}$$

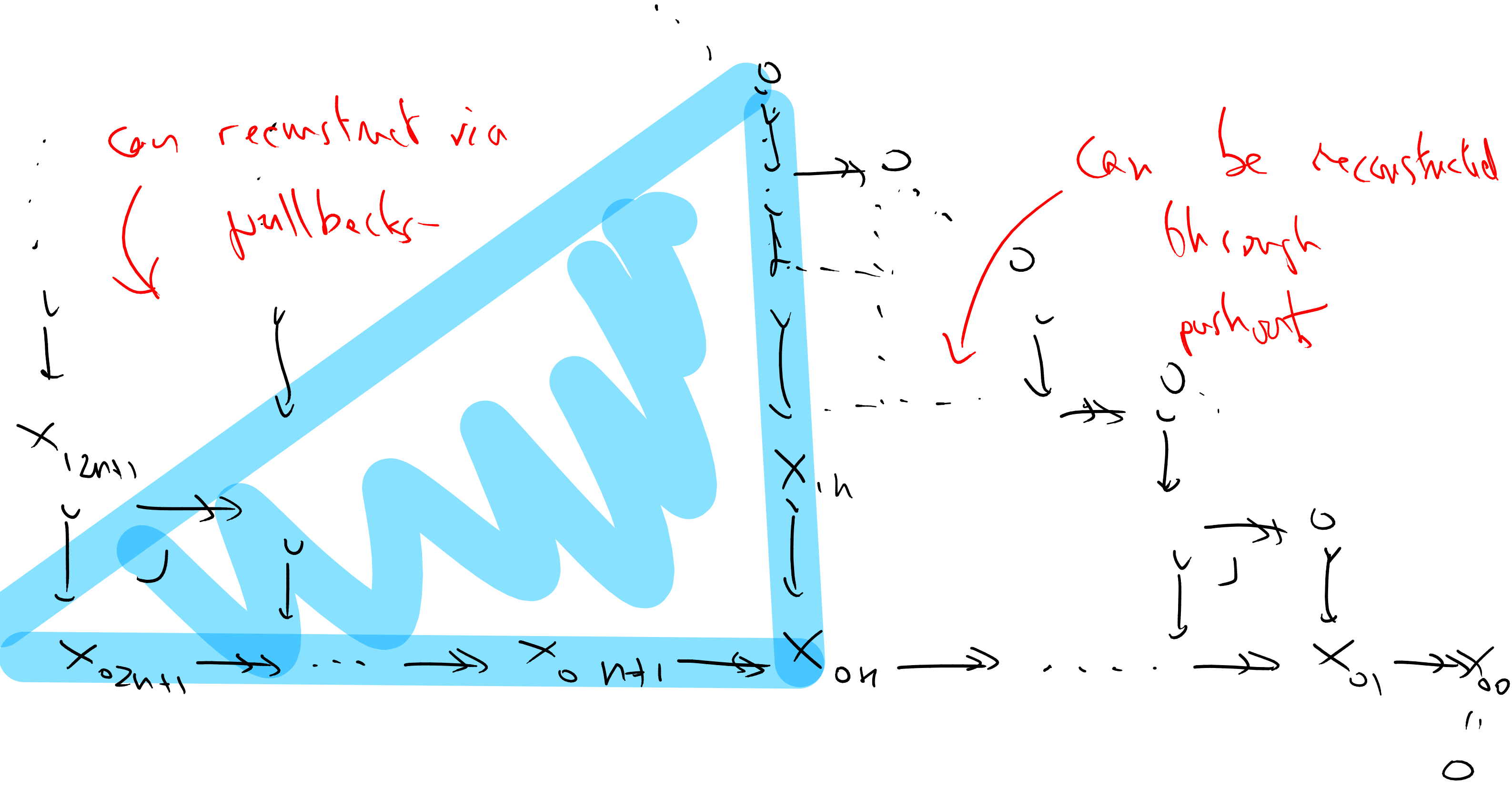
$$\mathrm{Fun}(O(\Delta^{n\,op} * \Delta^n), C) = \left(S_{2n+1}(C^{op})^{op}\right)^\simeq = \left(\varepsilon^* S_\bullet(C)\right)^\simeq_n$$

(here we use $\quad S(C) = S(C^{op})^{op}$

and that uses exactness)

Observation: Levelwise

$$S_{2n+1}(C^{op})^{op} \simeq \xrightarrow{\sim} Q_n(C)$$



$$\varepsilon^* S_\cdot(C) \simeq Q_\cdot(C)$$

To compare $S_\cdot(C)$ it is now sufficient to compare $\varepsilon^* S(C)$ and $S(C)$

Recall:

A quasi category $A_S \in \mathrm{Fun}(\Delta^{op}, S)$

via $\mathrm{Set} \to S$

$$CSS(A_S) \simeq A \qquad \varepsilon^* A_S \longrightarrow \varepsilon^* A \simeq \tilde{O}(A)$$

$$A_S \longrightarrow A \quad \text{"="} \quad \mathrm{Fun}(\Delta, A)^{\simeq}$$

$$CSS(\varepsilon^* A_S) \simeq \tilde{O}(A) \longrightarrow CSS(A_S) = A$$

Fact: $f : X \to Y$ (co)Cartesian fibration

s.t. $\forall y \in Y \quad X_y = f^{-1}(y)$ weakly contr.

$$\Rightarrow X[\mathcal{L}^{-1}] \simeq Y$$

$\mathcal{L} = \{$maps sent to identities$\}$

Example: $\tilde{O}(A) \to A$

$\tilde{O}(A) \to A^{op}$

$$CSS(\varepsilon^* \Delta^n) = \widetilde{\mathcal{O}}(\Delta^n) \longrightarrow \Delta^n$$

exhibits Δ^n as a localization of $CSS(\varepsilon^* \Delta^n)$

T simplicial space.

$$\mathcal{L}(T) \subset CSS(\varepsilon^* T)$$

of maps

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d, dashed, "\exists"] & \varepsilon^* T \arrow[dd] \\ \widetilde{\mathcal{O}}(\Delta^n) \arrow[ur, dashed] \arrow[d] & \\ \Delta^n \arrow[r, "\exists"'] & CSS(T) \end{tikzcd}$$

$= CSS(\varepsilon^* \Delta^n_J)$

s.t. $\Delta^1 \longrightarrow \Delta^n$ constant.

Prop. $\left[CSS(\varepsilon^* T)\right][\mathcal{L}(T)^{-1}] \xrightarrow{\sim} CSS(T)$

$(\Rightarrow)$ $\forall$ A ∞-cat seen as complete Segal space.

$$\mathrm{Map}(CSS(T), A) = \lim_{\Delta^n \to T} \mathrm{Map}(\Delta^n, A)$$

$$= \lim_{\Delta^n \to T} \mathrm{Map}(\left[CSS(\varepsilon^* \Delta^n)\right][\mathcal{L}(\Delta^n)^{-1}], A)$$

$$= \lim_{\Delta^n \to T} \mathrm{Map}_{\mathcal{L}(\Delta^n)}(\varepsilon^* \Delta^n, A) = \mathrm{Map}(CSS(T)[\mathcal{L}(T)^{-1}], A)$$

$\mathcal{L} \subset \varphi_{\cdot}(C)$ arrows =

$$\varepsilon^* S_{\cdot}(C) \simeq \varphi_{\cdot}(C)$$

$$\mathcal{L}(S_{\cdot}(C)) \leftrightsquigarrow \mathcal{L}$$

Thm $\varphi(C)[\mathcal{L}^{-1}] \simeq \mathrm{CSS}(S_{\cdot}(C))$

Cor. $K^{part}(C) \simeq \mathrm{Hom}_{\varphi(C)[\mathcal{L}^{-1}]}(0,0)$

Cor:

$$|\varepsilon^* S(C)| = |S(C)| = |\varphi(C)|$$