(p)-complete Frobenius and the action of $B\mathbb{N}$

Contents

Introduction

This note comes in four parts, modulo the appendices. The aim is to complement and explain the background material for [Yua19, 6]. The first part explains the theory of semi-orthogonal decompositions, [Lurc, 7.2.1], applied to (p)-completion of spectra. This will give us the formal properties needed. We also review some higher algebra from [Lurb, 7] which will be required when proving formal properties of p-adic thickenings. The second part recollects the proper Tate construction, defining F_p-stable and p perfect E_∞-rings. All examples given are p-perfect. The third part explains the integral action. The proof is rather formal, utilizing result from [Lurb, 4]. The last part explores a particular class of F_p-stable rings, the spherical Witt vectors. I would like to thank Allen, Harry, Marc and Peter; and special thanks to Denis for explaining the Segal conjecture.

1 (p)-completion

In this section, we clarify our basic objects of interest.

We apply the formalism in [Lurc, 7.3]. The set up there will usually consist of

- $\mathcal{C}$ a stable R-linear ∞-category. [1]
- R a connective $\mathbb{E}_2$-ring.

In particular, a stable R-linear category induces a map

$$\mathrm{LMod}_R \times \mathcal{C} \to \mathcal{C}$$

We will take $\mathcal{C} = \mathrm{Sp}, R = \mathbb{S}$, $I = (p) \subseteq \pi_0 \mathbb{S} \simeq \mathbb{Z}$.

Lemma 1.1. [Lurc, 7.3.1.4] We have a semi-orthogonal decomposition, reviewed in 1.1, on Sp

$$(\mathrm{Sp}^{\mathrm{loc(p)}}, \mathrm{Sp}^{(p)})$$

In particular, $\mathrm{Sp}^{(p)} \hookrightarrow \mathrm{Sp}$ has a left adjoint. We call $\mathrm{Sp}^{(p)}$ the ∞-category of (p)-*complete spectra*

Lemma 1.2. We have the adjunction[2], by [Lurc, 7.3.5.2],

$$\mathrm{Sp}^{(p)} \underset{\hookrightarrow}{\overset{\longleftarrow}{}} \mathrm{LMod}_{\mathbb{S}} \simeq \mathrm{Sp} \tag{1}$$

where the left adjoint inducing a symmetric monoidal structure on $\mathrm{Sp}^{(p)}$ [3] (property + structure), sending M to its (p)-completion $M_p\hat{}$

Remark 1.3. The tensor product on $\mathrm{Sp}^{(p)}$ is distinct from that in Sp and we denoted by $\hat{\otimes}$. Concretnly,

$$M \hat{\otimes} N \simeq (M \otimes N)_p\hat{}$$

Definition 1.4. We define the image of $\mathbb{S}$ under 1 to be $\mathbb{S}_p\hat{}$, the (p)-*complete sphere.*

Definition 1.5. We define the ∞-category of (p)-*complete* $\mathbb{E}_\infty$-*rings* as the pullback

$$\begin{array}{ccc} \mathrm{CAlg}^{(p)} & \longrightarrow & \mathrm{CAlg} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Sp}^{(p)} & \longrightarrow & \mathrm{Sp} \end{array}$$

Corollary 1.6. We have an adjunction induced from 1,

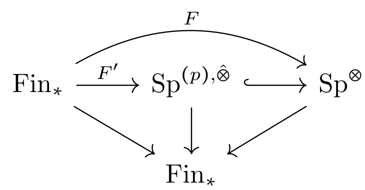
$$\mathrm{CAlg}(\mathrm{Sp}^{(p)}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{CAlg}$$

This restricts to an equivalence

$$\mathrm{CAlg}(\mathrm{Sp}^{(p)}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{CAlg}^{(p)} \tag{2}$$

[1] There are categories with left actions by LMod_R. See [Lurc, D.1.5].
[2] All adjunctions follow the Grothendieck convention, i.e. left adjoint on top
[3] Hence, the map itself is symmetric monoidal.

Proof. The induced adjunction is a corollary of [Lurb, 7.2.3.13]. To prove second equivalence, recall that E_∞-objects of a symmetric monoidal category $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ are sections preserving lifts of inert morphism, [Lurb, 2.1.2.7]. The problem is equivalently that: given



F factors through $\mathrm{Sp}^{(p),\hat{\otimes}}$ iff $F[1] \in \mathrm{Sp}^{(p)}$, where $[1] \in \mathrm{Fin}_*$. This follows from the fact that $\mathrm{Sp}^{(p)}$ is full and F is fully determined by its image at $[1]$. □

1.1 Semi-orthogonal decomposition and Bousefield localization

Let us elaborate the notion of semi-orthogonal decomposition in stable categories, following [Lurc, 7.2.1].

Definition 1.7. [Lurc, 7.2.1.1] Let $\mathcal{C}$ be a stable category. A pair $(\mathcal{C}_+, \mathcal{C}_-)$ of full subcategories is a *semi-orthogonal decomposition* if

1. $\mathcal{C}_+, \mathcal{C}_-$ are closed under finite limits and colimits, hence, stable.
2. For all $C_+ \in \mathcal{C}_+, C_- \in \mathcal{C}_-$, $\mathrm{Map}_{\mathcal{C}}(C, D) \simeq *$.
3. For all $C \in \mathcal{C}$, there exists a (co)fiber sequence

$$C_+ \to C \to C_-$$

where $C_+ \in \mathcal{C}_+, C_- \in \mathcal{C}_-$.

Remark 1.8. There can be potential naming confusion with the use of (*local*, *complete*) decompositions in [Lurc, 7.2.3] which corresponds to (*acyclic*,*local*) decomposition at a spectrum E in Sp.

Construction 1.9. Let $E \in \mathrm{Sp}$. A spectrum X is E-*acyclic* if

$$X \otimes E \simeq 0$$

The collection $\mathcal{C}_E$ of E-acyclic spectra is stable under shifts and colimits. By [Lurc, 7.2.1.4], we obtain a semi-orthogonal decomposition $(\mathcal{C}_E, \mathcal{C}_E^\perp)$ on Sp. The counit and unit of inclusions respectively, induces a functorial cofiber sequence

$$G_E X \to X \to L_E X$$

where $G_E X$ is E-acyclic and $L_E X$ is E-local. [4]

Remark 1.10. The functor $L_E : \mathrm{Sp} \to \mathcal{C}_E^\perp$ is classically referred to as the *Bousefiled localization* at E. The cofiber sequence also recovers the more familiar characterization of a localization $X \to L_E X$:

- The spectrum $L_E X$ is E-local.
- $X \to L_E X$ is an E-equivalence.

[4] We omitted writing the inclusion map.

Example 1.11. Let E be the discrete E_∞-ring spectrum $\mathbb{Q}$. Note that as $\mathbb{Q}$ is flat over $\mathbb{Z}$ in ordinary rings, and that $\pi_*\mathbb{S}$ is finite[5], from 1.3, $\mathbb{Q}$ is flat over $\mathbb{S}$ as $\mathbb{E}_\infty$-rings. Hence by 1.18, a spectrum X is E-acyclc iff

$$\pi_i(X \otimes \mathbb{Q}) \simeq \pi_i X \otimes \mathbb{Q} \simeq 0$$

i.e. $\pi_* X$ is no torsion free elements.

Definition 1.12. Now let us discuss the notion (p)-completion.

- The *Moore spectrum* with $\mathbb{Z}/p$ coefficients is the cofiber in Sp,

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \to \mathbb{S}/p$$

- Terminology in the literature. Taking $E = \mathbb{S}/p$ in 1.9, we say that a *spectrum X is (p)-complete* if it is (p)-local in the context of 1.9. We say it is an *p-adic equivalence* if it is an $\mathbb{S}/p$-equivalence.

1.2 Characterization for (x)-completion

We recall a useful description of the completion functor, [Lurc, 7.3.2]. As a corollary, we see that localization respects t-structure.

Proposition 1.13. [Lurc, 7.3.2.1] *Let R be an $\mathbb{E}_2$-ring. Let $\mathcal{C}$ be a stable R-linear ∞-category and let $x \in \pi_0 R$ be an element. For any object $C \in \mathcal{C}$, let $T(C)$ denote the limit of the tower*

$$\cdots \xrightarrow{x} C \xrightarrow{x} C \xrightarrow{x} C$$

The (x)-completion of C can be identified with the cofiber of the canonical map $\theta : T(C) \to C$.

Proof. By [Lurc, 7.3.1.4], it suffices to show have a fiber seqence

$$T(C) \to C \to \mathrm{cofib}(\theta)$$

A standard trick is to note that $\mathrm{cofib}(\theta)$ is given by $\Sigma\,\mathrm{fib}(\theta)$, from the diagram

$$\begin{array}{ccccc} \mathrm{fib}(\theta) & \longrightarrow & T(C) & \longrightarrow & 0 \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & \mathrm{cofib}(\theta) \end{array}$$

The only unproven step in the proof is that $T(C)$ is (x)-local. Again, [Lurc, 7.2.4.4] shows that we have a seminorthogonal deocmposition,

$$(\mathcal{C}^{\mathrm{nil}(x)}, \mathcal{C}^{\mathrm{loc}(x)})$$

Let $D \in \mathcal{C}^{\mathrm{nil}(x)}$. By definition, $D[x^{-1}] \simeq 0$.

$$\mathrm{Map}_{\mathcal{C}}(D, T(C)) \simeq \varprojlim \mathrm{Map}_{\mathcal{C}}(D, C) \simeq \mathrm{Map}_{\mathcal{C}}(\varinjlim D, \mathcal{C}) \simeq \mathrm{Map}_{\mathcal{C}}(D[x^{-1}], \mathcal{C}) \simeq *$$

where in the second to last line equivalence we realized $D[x^{-1}]$ as the colimit in $\mathcal{C}$, [Lurb, 7.2.3],

$$D \xrightarrow{x} D \xrightarrow{x} \cdots$$

$T(C)$ is thus (x)-local, from the semi-orthogonal decomposition. □

[5] Using Serre and Freudenthal's theorem

Corollary 1.14. [Lurc, 7.3.2.2] Under same conditions as 1.13, an object $C \in \mathcal{C}$ is (x) complete iff $T(C) \simeq 0$.

Corollary 1.15. [Lurc, 7.3.2.4] When $\mathcal{C} = \mathrm{LMod}_R$. Let R be an $\mathbb{E}_2$-ring, $x \in \pi_0 R$, $M \in (\mathrm{LMod}_R)_{\geq 0}$. Then $M_{(x)}\hat{} \in (\mathrm{LMod}_R)_{\geq 0}$.

Theorem 1.16. [Lurc, 7.3.6.9] Let R be a Noetherian $\mathbb{E}_\infty$-ring, and let $I \subseteq \pi_0 R$ be an ideal. Then the completion of $R_I\hat{}$ is flat over R.

1.3 Review of flatness and finiteness

We will be using various notion of finiteness and flatness. We recall the basic definitions here. For our purpose, in the statements below, we will be working with Noetherian $\mathbb{E}_k$-ring, for $k \geq 2$.

Definition 1.17. [Lurb, 7.2.2.10] Let M be a left module over $\mathbb{E}_1$-ring R. M is *flat* iff

- $\pi_0 M$ is flat over $\pi_0 R$ in usual sense.
- For each $n \in \mathbb{Z}$, $\pi_n R \otimes_{\pi_0 R} \pi_0 M \xrightarrow{\simeq} \pi_n M$.

As a consequence of the Tor-spectral sequence

Proposition 1.18. [Lurb, 7.2.2.13] *For each $n \in \mathbb{Z}$,*

$$\mathrm{Tor}_0^{\pi_0 R}(\pi_n M, \pi_0 R) \to \pi_n (M \otimes_R N)$$

is an isomorphism of abelian groups.

As a consequence of this one of the most important theorem is Lazard's characterization for module over connective $\mathbb{E}_1$-ring.

Theorem 1.19. [Lurb, 7.2.2.15] Let R be a connective $\mathbb{E}_1$-ring, and let N be a connective left A-module. The following are equivalent

1. The left R-module N can be obtained as a filtered colimt of projectiv elef tA-module.
2. The left R-module N is flat.
3. If M is a discrete right R-module, then $M \otimes_R N$.

Example 1.20. $\mathbb{S}_p\hat{} \otimes \mathbb{F}_p \simeq \mathbb{F}_p$. More generally, as $\otimes$ commutes with colimits, $(\bigoplus \mathbb{S}_p\hat{}) \otimes \mathbb{F}_p \simeq \bigoplus \mathbb{F}_p$.

Proof. As $\mathbb{S}_p\hat{}$ over $\mathbb{S}$, 1.16, by Lazard's theorem 1.19, their tensor is discrete. We also know that $\mathbb{S}_p\hat{}$ is connective, so by 5.14,

$$\pi_0(\mathbb{S}_p\hat{} \otimes \mathbb{F}_p) \otimes \mathbb{F}_p \simeq \mathbb{Z}_{(p)} \otimes \mathbb{F}_p \simeq \mathbb{F}_p$$

The latter follows by tensoring $\mathbb{Z}_{(p)}$ along the ses

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p$$

□

Definition 1.21. Let R be an $\mathbb{E}_1$-ring. $\mathrm{LMod}_R^{\mathrm{perf}}$ the *category of perfect R-modules*, be the smallest stable subcategory containing R and closed under retracts.

The are various characterizations, [Lurb, 7.2.4.2], in particular, they are the compact objects in LMod_R.

Example 1.22. Let $R = \mathbb{S}$, then $\mathbb{S}$ is perfect $\mathbb{S}$-module, being compact.

Example 1.23. $H\mathbb{Z}$ is not a perfect $\mathbb{S}$-module.

Definition 1.24. Let R be a connective $\mathbb{E}_k$-ring for $2 \leq k \leq \infty$. It is *Notherian* if

- it is *left coherent* as an $\mathbb{E}_1$-ring, i.e.
  1. $\pi_0 R$ is left coherent ring.[6]
  2. For each $n \geq 0$, the homotopy groups $\pi_m R$ is f.p. left module over $\pi_0 R$.
- $\pi_0 R$ is Noetherian.

Remark 1.25. For an $\mathbb{E}_k$-ring R, $k \leq 2 \leq \infty$, $\pi_0 R$ is a commutative $\mathbb{Z}$-algebra, [Lurb, 7.1.3].

Example 1.26. $\mathbb{S}$ is a Noetherian ring in $\mathrm{LMod}_{\mathbb{S}} \simeq \mathrm{Sp}$.

Example 1.27. $\mathbb{S}_p\hat{}$ is a Noetherian ring.

Proof. We have the following facts from classical theory

- $\mathbb{Z}_{(p)}$ is flat over $\mathbb{Z}$.
- $\mathbb{Z}_{(p)}$ is Noetherian. [7]

Result follows as flatness is determined by base change of the nth homotopy group base ring. □

In the setting of Noetherian modules, we have a nice characterization of *almost perfect* modules.

Proposition 1.28. [Lurb, 7.2.4.17] *Let R be a Noetherian ring. Then M is almost perfect iff*

- *M is bdd below.*
- *For all m, $\pi_m M$ is f.p. as a $\pi_0 R$ module.*

[6] Every finitely generated left ideal is finitely presented.
[7] It is a DVR.

2 F_p stable E_∞-rings

Definition 2.1. The proper Tate construction for G. For a finite group G, let

$$(-)^{\tau G} : \mathrm{Sp}^G \to \mathrm{Sp}$$

$$E \mapsto E^{\tau G} := \Phi^G(\beta X)$$

where β is the Borelification map (unit) from the adjunction

$$\mathrm{Fun}(BG, \mathrm{Sp}) \underset{\hookrightarrow}{\overset{\leftarrow}{\longrightarrow}} \mathrm{Sp}^G$$

Remark 2.2. The proper Tate construction is exact and is lax monoidal.

Proof. This properties are inhertied from β and Φ^G. □

Construction 2.3. [Yua19, 3.9] For a $G \twoheadrightarrow K$ a surj. of finite groups, $A \in \mathrm{CAlg}$, a map

$$\mathrm{can}_K^G : A^{\tau K} \to A^{\tau G}$$

Example 2.4. If $K = *, G = C_p$ we recover the standard canonical map.

In Peter's talk.

Theorem 2.5. [Yua19, 3.13] Let $\mathcal{Q} := h\,\mathrm{Fun}(\text{surj,inj, fin})$. There is an oplax monoidal functor

$$\Theta : \mathcal{Q} \to \mathrm{End}(\mathrm{CAlg}, \mathrm{CAlg})$$

It satisfies:

- Sends $G \in \mathcal{Q}$ to $(-)^{\tau G} : \mathrm{CAlg} \to \mathrm{CAlg}$.
- A left $(K \twoheadleftarrow G \to G)$ in $\mathcal{Q}$ in $\mathcal{Q}$ is sent to

$$\mathrm{can}_K^G : (-)^{\tau K} \to (-)^{\tau G}$$

- A right morphism $(H \leftarrow H \hookrightarrow G)$ in $\mathcal{Q}$ is sent to

$$\varphi_H^G : (-)^{\tau H} \to (-)^{\tau G}$$

Remark 2.6. The left, right, morphisms form a factorization of $\mathcal{Q}$.

Example 2.7. These maps recover the classical maps.

Definition 2.8. A spectrum X is F_p*-stable* if X is p-complete and for every finite dimensional $\mathbb{F}_p$ vector space V, the canonical map,

$$\mathrm{can}^V : X \to X^{\tau V}$$

is an equivalence.[8]

Definition 2.9. CAlg_p^F denotes the full subcategory of CAlg spanned by E_∞-rings whose *underlying* spectrum is F_p-stable.

[8] I.e. we are really inverting the equivalences see 2.14.

Example 2.10. Adams-Gunawardena-Miller. $\mathbb{S}_p\hat{}$ is F_p-stable.

Example 2.11. A non example. As $\tau C_p = tC_p$, $\mathbb{F}_p$ is *not* F_p stable. Since $\mathbb{F}_p$ is discrete by $\mathbb{F}_p^{tC_p}$ has nontrivial homotopy group in every degree.

Proposition 2.12. *F_p-stable spectra are closed under finite colimits.*

Proof. The proper Tate construction is given by $\Phi^G \beta X$. β is an exact functor, and Φ^G is exact too, by its definition. □

Corollary 2.13. Any spectrum which is finite over p-complete sphere is also F_p-stable.

Corollary 2.14. Θ restricts to a functor

$$\Theta : \mathcal{Q}\,\mathrm{Vect}_{\mathbb{F}_p} \to \mathrm{Fun}(\mathrm{CAlg}_p^F, \mathrm{CAlg}_p^F)$$

which induces an $\mathbb{E}_1$-monoidal morphism,

$$BK^{\mathrm{part}}\mathbb{F}_p \to \mathrm{Fun}(\mathrm{CAlg}_p^F, \mathrm{CAlg}_p^F)$$

*Proof. Step I. The map induced is in fact monoidal.*The first proposition follows by definition: by definition $(-)^{\tau V}$ fixes the F_p stable rings. With the oplax structure

$$(-)^{\tau U \oplus V} \to (-)^{\tau V} \circ (-)^{\tau U}$$

induces pointwise a commuting square

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{can}^V} & A^{\tau V} \\ \downarrow{\scriptstyle \mathrm{can}^{U\oplus V}} & & \downarrow{\scriptstyle (\mathrm{can}^U)^{\tau V}} \\ A^{\tau(U\oplus V)} & \longrightarrow & (A^{\tau U})^{\tau V} \end{array}$$

As A is F_p stable, the outer maps are equivalences, hence the bottom map is too. This implies Θ_p is monoidal rather than oplax monoidal.
Step II. Localize. *Little lemma:*[Yua19, 6.2]. For any surjection $V \twoheadrightarrow W$ of surjection can_W^V is also an equivalence:

$$\begin{array}{ccccc} & & X^{\tau W} & & \\ & \nearrow{\scriptstyle \mathrm{can}^W} & & \searrow{\scriptstyle \mathrm{can}_W^V} & \\ X & & \xrightarrow{\mathrm{can}^V} & & X \end{array}$$

Hence by combining [Yua19, 4.24.1, 8], and abstract nonsen [Lurb, 4.1.7.4] we obtain

$$BK^{\mathrm{part}}(\mathbb{F}_p) \simeq \mathcal{Q}\,\mathrm{Vect}_{\mathbb{F}_p}[\mathcal{L}^{-1}] \to \mathrm{End}(\mathrm{CAlg}_p^F)$$

□

2.1 p-perfect E_∞-rings

Definition 2.15. $A \in \mathrm{CAlg}$ is *p-perfect* if A is F_p stable and the Frobenius map

$$\varphi : A \to A^{tC_p}$$

is an equivalence. We let $\mathrm{CAlg}_p^{\mathrm{perf}}$ denote this category.

In Martin's talk we have proven:

Proposition 2.16. [Yua19, 6.11] *If $A \in \mathrm{CAlg}_p^{perf}$ for all fd vs V, the Frobenius map*

$$\varphi^V : A \to A^{\tau V}$$

is an equivalence.

Example 2.17. $\mathbb{S}_p\hat{}$ This is a consequence of Lin's theorem.

Example 2.18. Let $A \in \mathrm{CAlg}_p^{\mathrm{perf}}$ and X a finite space. Then A^X is p-perfect. This follows from formal properties: where we can reduce the case to $X = *$.

Example 2.19. Let R be a discrete perfect $\mathbb{F}_p$ algebra. Then $W^+(R)$ is p-perfect.

Once we have established the integral action in F_p-stable case, we obtain

Corollary 2.20. The monoidal functor

$$B\mathbb{Z}_{\geq 0} \to \mathrm{End}(\mathrm{CAlg}_p^{\mathrm{perf}})$$

factors through S^1.

3 Integral Action

Appendix A contains the background material for this section. Let us recall our aim. Let $\mathcal{A}$ be an ordinary additive category. Classically, to define a monoidal functor

$$\Phi : B\mathbb{Z}_{\geq 0} \to \mathrm{Fun}(\mathcal{A}, \mathcal{A})$$

of monoidal categories is equivalent to define a map of commutative monoids

$$\mathbb{Z}_{\geq 0} \to \mathrm{End}(\mathrm{id}_{\mathcal{A}})$$

In fact, it is sufficient to exhibit a map of monoids. This is because commutative monoids form a *full* subcategory of monoids, making commutative a *property.* Theorem A classically, is thus simple with $\mathcal{A} = \mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}$, discrete $\mathbb{F}_p$ algebras. The difference in the ∞-setting is that to exhibit we have to exhibit a map of $\mathbb{E}_2$-spaces, and this contains *structure.*

In Denis' talk.

Theorem 3.1. The natural map $K^{\mathrm{part}}(\mathbb{F}_p) \to \pi_0 K^{\mathrm{part}}(\mathbb{F}_P) \simeq \mathbb{Z}_{\geq 0}$ is an isomorphism in $\mathbb{F}_p$-homology.

Theorem 3.2. Theorem A. There is a map of $\mathbb{E}_1$-monoidal categories

$$B\mathbb{Z}_{\geq 0} \to \mathrm{Fun}(\mathrm{CAlg}^F_p, \mathrm{CAlg}^F_p)$$

$$n \mapsto \varphi^n : \mathrm{id} \to \mathrm{id}$$

Proof. By 2.14 we a morphism in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Cat})$,

$$BK^{\mathrm{part}}\mathbb{F}_p \to \mathrm{Fun}(\mathrm{CAlg}^F_p, \mathrm{CAlg}^F_p)$$

As the truncation functor $\mathcal{S} \to \mathcal{S}_{\leq 0}$ is symmetric monoidal, we obtain a morphism in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{S})$,

$$K^{\mathrm{part}}\mathbb{F}_p \to \pi_0 K^{\mathrm{part}}\mathbb{F}_p \simeq \mathbb{Z}_{\geq 0}$$

In particular, this is a morphism in $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$. Applying the functor

$$\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}) \underset{\Omega}{\overset{B}{\rightleftarrows}} \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Cat}_\infty) \tag{3}$$

The claim is that we may find a lift to the follow diagram

$$\begin{array}{ccc} BK^{\mathrm{part}}(\mathbb{F}_p) & \longrightarrow & \mathrm{Fun}(\mathrm{CAlg}^F_p, \mathrm{CAlg}^F_p) \\ \downarrow & \nearrow & \\ B\mathbb{Z}_{\geq 0} & & \end{array}$$

Step I. Reducing to a problem of $\mathbb{E}_2$*-space.* Now we apply the adjunction 3, This is equivalent to providing a lift

$$\begin{array}{ccc} K^{\mathrm{part}}(\mathbb{F}_p) & \longrightarrow & \mathrm{End}(\mathrm{id}_{\mathrm{CAlg}^F_p}) \\ \downarrow & \nearrow & \\ \mathbb{Z}_{\geq 0} & & \end{array}$$

or an equivalence

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(\mathbb{Z}_{\geq 0}, \mathrm{End}(\mathrm{id}_{\mathrm{CAlg}_p^F})) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(K^{\mathrm{part}}(\mathbb{F}_p), \mathrm{End}(\mathrm{id}_{\mathrm{CAlg}_p^F})) \tag{4}$$

Step Ia. Reduce to a problem in $\mathcal{S}$ via monadic adjunction. Hence, of (p)-completion. In 5.7, an $\mathbb{E}_2$ space X, is equivalent $|\mathrm{Bar}_T(T,X)_\bullet|$ with T being induced from monadic adjunction

$$U : \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}^\times) \to \mathcal{S}$$

This is the classical resolution for a topological monoid. For any other $\mathbb{E}_2$-space Y, $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}}(X,Y)$ is a geometric realization of

$$\varprojlim_n \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})}(T^n X, Y) \simeq \varprojlim_n \mathrm{Map}_{\mathcal{S}}((X)^n, Y)$$

As $\mathbb{F}_p$ homology isomoprhism commutes with products, we reduce to showing $\mathrm{End}(\mathrm{id}_{\mathrm{CAlg}_p^F})$ is (p)-complete space.

Step II. Reduction of $\mathrm{End}(\mathrm{id}_{\mathrm{CAlg}_p^F})$ *to mapping space in spectra* By the ∞-version of end formula, 5.11, $\mathrm{End}(\mathrm{id}_{\mathrm{CAlg}_p^F})$ may be written as limit of spaces

$$\varprojlim \mathrm{Map}_{\mathrm{CAlg}_p^F}(A,B) \simeq \varprojlim \mathrm{Map}_{\mathrm{CAlg}}(A,B)$$

where A, B are (p)-complete $\mathbb{E}_\infty$-rings. The equivalence follows from 1.5, i.e. that (p)-complete rings form a full subcategory. As in *Step III*, we can reduce a problem of mapping space of algebras to a limit of underlying spectra, i.e.

$$\mathrm{Map}_{\mathrm{CAlg}}(A,B) \simeq \varprojlim \mathrm{Map}_{\mathrm{Sp}}(A^{\otimes n}, Y)$$

Step IVa. Proving mapping space is p-complete. We show: for any spectra A, and p-complete spectra B, $\Omega^\infty \mathrm{map}_{\mathrm{Sp}}(A,B)$ is p-complete. Result then follows as p-complete spaces are closed under limits.

Let $X \to Y$ be a p equivalence in spaces. Then it suffices to show equivalence,

$$\mathrm{Map}_{\mathcal{S}}(Y, \Omega^\infty \mathrm{map}_{\mathrm{Sp}}(A,B)) \longrightarrow \mathrm{Map}_{\mathcal{S}}(X, \Omega^\infty \mathrm{map}_{\mathrm{Sp}}(A,B))$$

By adjunction, we may rewrite the LHS as

$$\mathrm{Map}_{\mathcal{S}}(Y, \Omega^\infty \mathrm{map}_{\mathrm{Sp}}(A,B)) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ Y, \mathrm{map}_{\mathrm{Sp}}(A \otimes B)) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ Y \otimes A, B)$$

and similarly for the the RHS. By 5.17, $\Sigma^\infty_+ X \to \Sigma^\infty_+ Y$ is a p-adic equivalence. Thus, by definition of B being p-complete, result follows. □

Remark 3.3. There is also a variant of this theorem for global algebra, [Yua19, 6.9].

4 Spherical Witt Vectors

Appendix B contains the background materials for this section. In this section, we first review the theory of Witt vectors in the spectral setting, following [Lurd, 5.2]. This is a generlization of the theory of p-adic thickenings. A nice summary of classical references is in [Sch11, 5.11,13].

Definition 4.1. [Lurd, 5.2.1] Let $A \in \mathrm{CAlg}_{\geq 0}$. $I \subseteq \pi_0(A)$ a f.g. ideal, $A_0 := \pi_0(A)/I$. Given a commutative diagram σ in $\mathrm{CAlg}_{\geq 0}$,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

*σ exhibits f as an A-thickening of f_0*If the following conditions are satisfied

1. The $\mathbb{E}_\infty$-ring B is I-complete.
2. The diagram σ induces an isomoprhism of commtuative rings $\pi_0 B/I\pi_0 B$ with B_0.
3. Let $R \in (\mathrm{CAlg}_{A,\geq 0})_{\hat{I}}$, Then

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \simeq \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$$

There is a very general existence result for thickenings, [Lurd, 5.2.5]. This is a problem of checking finiteness condition on A_0. When A is Noetherian, by 1.28, conditions are trivially satisfied.

Theorem 4.2. [Lurd, 5.2.5] Let A be a Noetherian $\mathbb{E}_\infty$-ring, with $I \subseteq \pi_0 A$ a f.g. ideal such that $A_0 := \pi_0 A/I \simeq \mathbb{F}_p$. Let $A_0 \xrightarrow{f_0} B_0$ be any discrete perfect $\mathbb{F}_p$ algebra. Then there exists a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \xrightarrow{f_0} & B_0 \end{array}$$

which exhibits f as an A-thickening of f_0. Moreover, the square is a pushout.

Proposition 4.3. [Lurd, 5.2.11] *In 4.2, if A is p-complete, A_0 and B_0 are Noetherian and that $f_0 : A_0 \to B_0$ is flat, then $f : A \to B$ is flat.*

Corollary 4.4. In 4.2, B is a flat A-algebra.

Proof. Compostion of flat morphisms is flat. The morphism $A_{\hat{p}} \to B$ is flat by 4.3, and $A \to A_{\hat{p}}$ by 1.16. ☐

The handiness of flatness comes when classifying the lift.

Case study 4.5. We will take our B_0 to be any perfect $\mathbb{F}_p$ algebra.

(X) $A = \mathbb{Z}$, $I = p\mathbb{Z}$, and its p-adic counterpart, $A = \mathbb{Z}_{(p)}, I = p\mathbb{Z}_{(p)}$.

(Y) $A = \mathbb{S}$, $I = p\mathbb{Z}$, and its p-adic counterpart, $A = \mathbb{S}_{\hat{p}}, I = p\mathbb{Z}_{(p)}$.

- By universal property of completion, whether we lift by the p-completion version, we obtain the same result.
- In case (X), the p-adick thickening $W(B_0)$ is a flat $\mathbb{Z}$-algebras, in particular, discrete. By 4.2,
  1. $W(B_0)$ is a discrete $\mathbb{E}_\infty$-ring which is (p)-complete.
  2. $\pi_0 W(B_0)/p\pi_0 W(B_0) \simeq B_0$.
  3. For any p-complete $\mathbb{E}_\infty$ connective $\mathbb{Z}$-algebra R,

  $$\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Z}}}(W(B_0), R) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(B_0, \pi_0 R/p\pi_0 R)$$

  When we plug in discrete R algebras in 3. we recover *ring of Witt vectors*.
- In case (Y), the lifts are flat $\mathbb{S}$-algebra, $W^+(B_0)$. We call such unique lifts $W^+(B_0)$ the *Spherical Witt vectors* of B_0. Enjoying similarl universal properties.
- We have $\pi_0\mathbb{S} \to \pi_0 W^+ B_0$ exhibiting a lift of (X). So $\pi_0 W^+ B_0 \simeq W(B_0)$.

Example 4.6. $W(\mathbb{F}_p) \simeq \mathbb{Z}_{(p)}$ and $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p\hat{}$.

Corollary 4.7. Let B_0 be a discrete perfect $\mathbb{F}_p$-algebra, then $W^+(B_0)$ can be exhibited as a p-completion of

$$\bigoplus \mathbb{S}_p\hat{}$$

Proof. Firstly, we have the equivalence,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}_p\hat{}}((\bigoplus \mathbb{S}_p\hat{})_p\hat{}, W^+(B_0)) &\simeq \mathrm{Map}_{\mathrm{CAlg}}(\bigoplus \mathbb{S}_p\hat{}, W^+(B_0)) \\
&\simeq \prod \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}_p\hat{}, W^+(B_0)) \\
&\simeq \prod \mathrm{Map}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(\mathbb{F}_p, B_0)
\end{aligned}$$

We remark in order for each equivalence

- First one is by universal property of p-completion.
- Second is by universal property of coproduct.
- Third one is by universal property of $\mathbb{S}_p\hat{} \simeq W^+(\mathbb{F}_p)$.

In otherwords, we can exhibit an equivalence by choosing basis of B_0. ☐

Our goal is to prove the following.

Theorem 4.8. [Yua19, 6.6] Let R be a discrete perfect $\mathbb{F}_p$-algebra. Then spectrum $W^+(R)$ is F_p-stable.

Proof. We will establish that (1) is a p-completion and (2) is an equivalence in the following diagram

$$\begin{array}{ccccc}
\oplus_\alpha \mathbb{S}_p\hat{} & \longrightarrow & (\oplus \mathbb{S}_p\hat{})_p\hat{} \simeq W^+(R) & & \\
 & & & \searrow & \\
\downarrow \simeq & & & & ((\oplus \mathbb{S}_p\hat{})_p\hat{})^{\tau V} \\
 & & & \nearrow^{(2)} & \\
\oplus(\mathbb{S}_p\hat{})^{\tau V} & \xrightarrow{(1)} & (\oplus \mathbb{S}_p\hat{})^{\tau V} & &
\end{array} \tag{5}$$

Result then follows, say since equivalence of p-complete objects can be checked by $- \otimes \mathbb{S}/p$. We first prove 4.11, which states 5 holds for τ replaced by t. ☐

4.1 Proof of Tate version

This follows from two steps. We will need the following special case of Segal's conjecture.

Theorem 4.9. Segal's conjecture at $\mathbb{S}$. Suppose G is a p-group. We have a p-adic equivalence

$$\bigoplus_{(H)\neq e} (\Sigma^\infty (S^0)^H)_{hWH} \xrightarrow{\simeq} (\Sigma^\infty S^0)^{tV}$$

where

- S^0 has trivial G-action.
- The genuine H-fixed points of a space is the restriction functor induced from the inclusion of the full subcategory spanned by G/H in $\mathcal{O}_G$.

  $$BW_G H \hookrightarrow \mathcal{O}_G$$

  This induces a functor

  $$\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S}_*) \to \mathrm{Fun}(BW_G H, \mathcal{S}_*)$$

We note that the lhs can be equivalent written as

$$\bigoplus_{(H)\neq e} (\Sigma^\infty S^0) \otimes BW_G H \simeq \bigoplus_{(H)\neq e} \Sigma^\infty_+ BW_G H$$

Corollary 4.10. The homotopy groups of $(\mathbb{S}/p)^{tV}$ are finite.

*Proof. Step I. Reducing the problem via Segal's conjecture.*As smashing with $\mathbb{S}/p$ commutes with Tate map,

$$(\mathbb{S}/p)^{tV} \simeq \mathbb{S}^{tV} \otimes \mathbb{S}/p \simeq \bigoplus_{(H)\neq e} \Sigma^\infty_+ BW_G H \otimes \mathbb{S}/p$$

By Segal's conjecture. As there are only finitely many terms on the rhs, it suffices to show for all G

$$\Sigma^\infty_+ BG \otimes \mathbb{S}/p$$

has finite homotopy groups.

Step II. Computing the homotopy groups. We apply the *Atiyah Hirzebruch spectrall sequence (AHSS)*

$$E^2_{s,t} \coloneqq H_s(X, \pi_t(E)) \Rightarrow \pi_{s+t}(X \otimes E)$$

applied to the special case $X = \Sigma^\infty_+ BG$, $E = \mathbb{S}/p$. We obtain

$$E^2_{s,t} \coloneqq H_s(BG, \pi_t(\mathbb{S}/p)) \Rightarrow \pi_{s+t}(\Sigma^\infty_+ BG \otimes \mathbb{S}/p)$$

We make two observations.

- For each $n = s + t$, there are only finitely many contributions, since we must have both $s, t \geq 0$.
- $\pi_t(\mathbb{S}/p)$ is finite. This follows from Serre's theorem and the les associated to the sequence

  $$\mathbb{S} \xrightarrow{p} \mathbb{S} \to \mathbb{S}/p$$

It suffices to show homology of BG is finite for any abelian group M. But this is true as BG has only finitely man cells in each dimension. □

Proposition 4.11. *Let I be a set and let V a finite dimensional $\mathbb{F}_p$ vector space.*

1. *The natural map*

$$\bigoplus_{\alpha}(S_p\hat{})^{tV} \to (\bigoplus_{\alpha} S_p\hat{})^{tV}$$

is a p-completion.

2. *The natural map*

$$(\bigoplus_{\alpha\in I} S_p\hat{})^{tV} \to ((\bigoplus_{\alpha\in I} S_p\hat{})_p\hat{})^{tV}$$

is an equivalence.

Proof. Step I. Reduce problem with smashing Moore spectrum. By 6.3 and definition of p-completion, it suffices to show both maps are equivalences after smashing with Moore spectrum. Indeed, for a geneeral spectrum X, there is functorial orthogonal factorization

$$G_{\mathbb{S}/p}X \to X \to X_p\hat{}$$

to a $\mathbb{S}/p$-acyclic spectrum $G_{\mathbb{S}/p}X$, and a p-complete spectrum, $X_p\hat{}$. This implies

A $X \to X_p\hat{}$ is an equivalence after smashing with $\mathbb{S}/p$.

B From the diagram

$$\begin{array}{ccccc} G_{\mathbb{S}/p}X & \longrightarrow & X & \longrightarrow & X_p\hat{} \\ \downarrow & & \downarrow & & \downarrow \\ G_{\mathbb{S}/p}Y & \longrightarrow & Y & \longrightarrow & Y_p\hat{} \end{array}$$

If X, Y are both p-complete, and they induce an equivalence after smashing with $\mathbb{S}/p$, by 2-3 law, $X \to Y$ is an equivalence.

Step Ia. Reduce case to 1. and $\mathbb{F}_p$. In the case of 2. By 6.2, and observation A, the map is an equivalence. In the case of 1, we observe the fiber sequence

$$G_{\mathbb{S}/p}\mathbb{S} \to \mathbb{S} \to \mathbb{S}_p\hat{}$$

of p-completion implies that

$$\mathbb{S}/p \to \mathbb{S}_p\hat{} \otimes \mathbb{S}/p$$

Thus as smashing commutes with Tate construction and coproducts, we are reduced to proving equivalence of

$$\bigoplus_{\alpha}(\mathbb{S}/p)^{tV} \to (\bigoplus_{\alpha}\mathbb{S}/p)^{tV}$$

Step II. Prove that $\pi_i(\tau_{\leq n}\mathbb{S}/p)^{tV}$ is pro-constant with value $\pi_i(\mathbb{S}/p)^{tV}$. By [NS18, III.1.8], it suffices to show with a sequence of finite sets whose limit is finite. This boils down to showing $(\mathbb{S}/p)^{tV}$ has finite homotopy groups. We will omit the proof of Step II first.

Step III. Prove that Step II commutes with inifinite direct sum. $\{\pi_i(\tau_{\leq n}(\bigoplus \mathbb{S}/p))^{tV}\}_n \simeq \pi_i(\bigoplus \mathbb{S}/p)^{tV}$.
Step IIIa. Apply direct sum. The functor $A \mapsto \bigoplus_\alpha A$ is left exact between abelian groups. As pro-isomorphism are absolute under left exact functors, [9]

$$\{\pi_i(\bigoplus(\tau_{\leq n}\mathbb{S}/p)^{tV})\}_n \simeq \pi_i(\bigoplus(\mathbb{S}/p)^{tV}) \tag{6}$$

where we used that π_i commtues with coproducts.

Step IIIb. Bringing $\bigoplus$ into tV. It follows from 4.13, exactness of $(-)_{hG}, (-)^{hG}$, that tV commtues with coproduct in this case. Further, $\tau_{\leq n} : \mathrm{Sp} \xrightarrow{\tau_{\leq n}} \mathrm{Sp}_{\leq n} \hookrightarrow \mathrm{Sp}$ commutes with filtered colimits (and is exact) hence commutes with coproducts. Hence, the system in lhs, is in fact

$$\{\pi_i((\tau_{\leq n}\bigoplus \mathbb{S}/p)^{tV})\}_n \simeq \pi_i(\bigoplus(\mathbb{S}/p)^{tV}) \tag{7}$$

Step IIIc. Apply Milnor $\lim^1$. The particular sequence we apply to is

$$\cdots \to (\tau_{\leq n}\bigoplus \mathbb{S}/p)^{tV} \to \cdots \to (\tau_{\leq 0}\bigoplus \mathbb{S}/p)^{tV}$$

Thus, we may commute the π_i with $\varprojlim$. Hence, we obtain

$$\pi_i(\bigoplus \mathbb{S}/p)^{tV} \simeq \pi_i \varprojlim_n(\tau_{\leq n}\bigoplus \mathbb{S}/p)^{tV} \simeq \pi_i(\bigoplus(\mathbb{S}/p)^{tV})$$

using that Tate construction commutes with Postnikov for the first equivalence. ☐

Corollary 4.12. $(\bigoplus \mathbb{S}_p\hat{})^{tV}$ is connective.

Proof. Step I. Reduce to the case of one indexing set. As completion of an $\mathbb{S}$-module remains connective, by 1. of 4.11, it suffices to show the case for $(\mathbb{S}_p\hat{})^{tV}$.

Step II. Segal conjecutre is p-completion. We know $\mathbb{S}^{tV}$ is p-adic equivalent $(\mathbb{S}_p\hat{})^{tV}$. Since $\mathbb{S}^{tV}$ is p-complete, Segal's conjecture, 4.9 implies it is a (p)-completion of connective spectra, which is connective. ☐

In the above we have used the following lemma.

Lemma 4.13. Homotopy fixed points commuting of filtered colimits. For G a finite grp, $\{X_i\}_{i\in I}$ a uniformly bounded above filtered colimit of G equivariant spectra. Then

$$(\varinjlim X_i)^{hG} \simeq \varinjlim(X_i)^{hG}$$

4.2 Proof of proper Tate version

Corollary 4.14. Let $X \in \mathrm{Sp}^G$, $X^{\tau G}$ fits into a cofiber sequence

$$\mathrm{colim}_{\mathcal{E}}((X)^{hH})_{hWH} \to X^{hG} \to X^{\tau G}$$

for some contractible category $\mathcal{E}$.

Proposition 4.15. *Let I be a set and let V a finite dimensional $\mathbb{F}_p$ vector space.*

[9] we have slept under the rug of the indexing of n.

1. *The natural map*
$$\bigoplus_{\alpha}(S^{p})^{\tau V} \to (\bigoplus_{\alpha} S_p^{\wedge})^{\tau V}$$
is a p-completion.

2. *The natural map*
$$(\bigoplus_{\alpha\in I} S_p^{\wedge})^{\tau V} \to ((\bigoplus_{\alpha\in I} S_p^{\wedge})_p^{\wedge})^{\tau V}$$
is an equivalence.

Proof. Step I. Reducing the problem to Tate construction. We claim that by 4.14 we have the following diagram of functors

$$\begin{array}{ccccc} (-)_{hV} & \longrightarrow & (-)^{hV} & \longrightarrow & (-)^{tV} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{colim}_{W\subsetneq V\in\mathcal{E}}((-)^{hW})_{hV/W} & \longrightarrow & (-)^{hV} & \longrightarrow & (-)^{\tau V} \\ \downarrow & & \downarrow & & \downarrow \\ F := \operatorname{colim}_{W\subsetneq V}((-)^{tW})_{hV/W} & \longrightarrow & * & \longrightarrow & \Sigma F \end{array}$$

where each column and row are cofiber. To obtain the formula for F we note that the indexing category $\mathcal{E}$ in 4.14 is contactiblle. Hence we rewrite

$$(-)_{hV} \simeq \operatorname{colim}_{\mathcal{E}}((-)^{hW})_{hV/W}$$

Result then follows from exactness of the functors involved. How we will apply this diagram is to show that every thing can be reduced to checking for

$$((-)^{tW})_{hV/W}$$

Step II. The properties of map. In the case of (2) it suffices to show that ΣF and $(-)^{tV}$ satisfies the property

$$\bigoplus \mathbb{S}_p^{\wedge} \to (\bigoplus_{\alpha} \mathbb{S}_p^{\wedge})_p^{\wedge} \text{ are mapped to equivalence}$$

This is reduced to show that the map $((-)^{tW})_{hV/W}$ satisfies the property. Similarly, for the equivalence (1) after smashing $\mathbb{S}/p$. In this case we will need the further fact that $h_{V/W}$ commutes with infinite direct sums.

Step IIIa. Prove that $\left(\oplus \mathbb{S}_p^{\wedge}\right)^{\tau V}$ *is p-complete.* As p-complete spectra is closed under extension and finite colimits, it suffices to prove the case for

$$((\bigoplus_{\alpha} \mathbb{S}_p^{\wedge})^{tW})_{hV/W} \simeq (\bigoplus \mathbb{S}_p^{\wedge})^{tW} \otimes BV/W$$

Note that the the action is trivial: this is true in particular when the sequence

$$0 \to W \to V \to V/W$$

splits. But this holds as we are working with $\mathbb{F}_p$-vector spaces.

By characterization of (p)-completeness, this is equivalent to checking the inverse limit

$$\cdots \xrightarrow{p} (\bigoplus \mathbb{S}_p^{\wedge})^{tW} \otimes BV/W \xrightarrow{p} (\bigoplus \mathbb{S}_p^{\wedge})^{tW} \otimes BV/W \xrightarrow{p} (\bigoplus \mathbb{S}_p^{\wedge})^{tW} \otimes BV/W$$

is zero.

Step IIIb. As have proven as corollary, $(\bigoplus \hat{\mathbb{S}_p})^{tW}$ is ocnnective. Now each homotopy group $\pi_k(\varprojlim C_i)$ depends only on π_k, π_{k-1} of each term. Observe then we have the exact sequence

$$\mathrm{sk}_L\, BV/W \to BV/W \to \mathrm{cofib}(f_L)$$

where the cofiber is $L+1$-connective. So by les we have that for L sufficiently large, we have

$$\pi_k(A \otimes \mathrm{sk}_L\, BV/W) \simeq \pi_k(A \otimes BV/W)$$

this follows as if $A \in \mathrm{Sp}_{\geq b}, \mathrm{cofib}(f_L) \in \mathrm{Sp}_{\geq k+1}$ then their tensor lies in $\mathrm{Sp}_{\geq b+k+l}$. Hence

$$\pi_k(A \otimes \mathrm{cofib}\, f_L) \simeq 0$$

□

5 Appendix A

5.1 Free resolution for $\mathcal{O}$-algebra objects

In this section, we explain how we can construction a free resolution for algebra objects.

Context 5.1. We state the context here

- Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, [Lurb, 2.1].
- Denote $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ as the *$\mathcal{O}$-algebra objects of* $\mathcal{C}$, the full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by fibered maps which preserves inert morphisms.

Remark 5.2. The reason for the notation is that this is distinct to the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ defined in [Lurb, 2.1.2.7]. These coincides when $\mathcal{O}^{\otimes} = \mathrm{Fin}_*$.

Proposition 5.3. [Lurb, 3.2.3] *The canonical forgetful functor*

$$U : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$$

satisfies the following properties.

1. *U admits a left adjoint.*
2. *U is conservative.*
3. *Suppose that the fibration of ∞-operads is compatible with κ-small sifted simplicial sets, then G commutes with κ-small sifted colimits. In particular, with κ-small geometric realizations.*

Proof. 1. Follows from [Lurb, 3.1.3.5]. 2. follow from [Lurb, 3.2.2.6]. 3. follows from [Lurb, 3.2.2.3]. □

Now we may apply the ∞-Barr Beck Theorem.

Theorem 5.4. [Lurb, 4.7.3.5] Let $G : \mathcal{D} \to \mathcal{C}$ be a functor which admits a left adjoint. The following are equivalent

1. The functor G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.
2. There exists a an $\mathbb{E}_1$-monoidal category $\mathcal{E}^{\otimes}$ which acts on $\mathcal{C}$ by the left, and an algebra object $A \in \mathrm{Alg}(\mathcal{E})$ exhibiting commuting diagram
$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\simeq} & \mathrm{LMod}_A(\mathcal{C}) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle U} & \\ & \mathcal{C} & \end{array}$$
3. The functor G satisfies the following conditions:
   - The functor G is conservative.
   - Everty G-split simplicial object admits a colimit in $\mathcal{D}$ and is preserved by G.

Remark 5.5. Going from 1 to 2 is by definition of monadic. Here we take $\mathcal{E}^{\otimes} = \mathrm{Fun}(\mathcal{C}, \mathcal{C})^{\otimes}$, $A = T \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))$ to be the endomorphism monad, [Lurb, 4.7.3.2].

The hard part is to go from 3 to 1. For this we utilize the notion of Bar construction, [Lurb, 4.4.2].

Corollary 5.6. [Lurb, 4.7.3.14] Let $G : \mathcal{D} \to \mathcal{C}$ be a monadic functor. Then for every object $D \in \mathcal{D}$, there exists a G-split simplicial object having colimit D, such that each D_n lies in the essential image of F.

Proof. By equivalence of 1 and 2, it suffices to prove the case for $\mathrm{LMod}_T(\mathcal{C})$. This follows from from the bar construction, [Lurb, 4.4.2.7] and remark [Lurb, 4.7.2.7] that

$$|\mathrm{Bar}_T(T, M)_\bullet| \simeq M$$

□

Corollary 5.7. Every object $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ is the geometric realization of a simplicial object in $A_\bullet$, where each A_n is free, i.e. in the essential image of left adjoint.

Proof. Hypothesis of 5.4 are satisfied for the adjunction 5.3. Now apply 5.6. □

5.2 Ends and coends

We define the notion of *end*. Let us recall *twisted arrow* ∞*-categories*. There are various places for details of this construction, for example [Lurb, 5.2], [BG16, 1].

Definition 5.8. Let X be a small category, and $\tilde{\mathcal{O}}(X) \to X^{op} \times X$ denote a *twisted category* of X.

Remark 5.9. For an category X, this is the left fibration classifed by the functor

$$X^{op} \times X \to \mathcal{S}$$

$$(c, d) \mapsto \mathrm{Map}_X(c, d)$$

Definition 5.10. Let $T : X^{op} \times X \to \mathcal{C}$ be a functor, then the *end* is given by

$$\int_X T := \lim_{\tilde{\mathcal{O}}(X)} T \circ \tilde{\mathcal{O}}(X) \tag{8}$$

Proposition 5.11. [Gla15, 2.3] *Let $\mathcal{D}$ be any category, $F, G : X \to \mathcal{D}$ be functors. Then*

$$\int_X F^{op} \times G \simeq \mathrm{Map}_{\mathrm{Fun}(X,\mathcal{D})}(F, G)$$

5.3 Hurewicz for (p)complete rings

Here we review a classical piece of spectral sequence in the spectral setting, [Lurb, 7.2.1]. In this discussion, we always let R to be an $\mathbb{E}_1$-ring. Here we outline the consturction

Construction 5.12. [Lurb, 7.2.1.18] Let $M \in \mathrm{RMod}_R, N \in \mathrm{RMod}_R$. Let S be the collection of all left R-modules of the form $R[n]$ where n is an integer.

- We obtain a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 2}$ associated to the simplicial spectrum $M \otimes_R P_\bullet$ where $P_\bullet$ is an S-free S-hypercovering of N.

If we unravel this, we observe that

1. The normalized chain complex associated to $\pi_* P_\bullet$ is a resolution of $\pi_* N$ by garded free left $\pi_* R$-modules.
2. We have canonical isomorphism
$$E_2^{p,*} \simeq \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N) \tag{9}$$

Lemma 5.13. If $N \in (\mathrm{LMod}_R)_{\geq n}$, then we can find a quasi-free resolution $P_\bullet$ with $P_n \in (\mathrm{LMod}_R)_{\geq n}$.

Corollary 5.14. Let R be a connective $\mathbb{E}_1$-ring, $M \in (\mathrm{LMod}_R)_{\geq k}, N \in (\mathrm{LMod}_R)_{\geq l}$ then

- $M \otimes_R N \in (\mathrm{LMod}_R)_{\geq k+l}$.
- $\pi_{k+l}(M \otimes_R N) \simeq \pi_k M \otimes_{\pi_0 R} \pi_l N$.

Proof. Let us take a resolution of N as given by from 5.13. In otherwords, we have

$$\cdots \to \pi_* P_m \to \cdots \to \pi_* P_0 \to \pi_* N$$

of graded $\pi_* R$ modules. And $E_2^{p,*}$ is given by taking pth homology of the following complex,

$$\cdots \to \pi_* M \otimes_{\pi_* R} \pi_* P_m \to \cdots \to \pi_* M \otimes_{\pi_* R} \pi_* P_0 \to 0$$

In each graded module, the first nonzero term occurs at

$$E_2^{0,k+l} \simeq \pi_k M \otimes_{\pi_0 R} \pi_l N$$

□

Corollary 5.15. Hurewicz Theorem. Let $X \in (\mathrm{Sp})_{\geq k}$. Then

- $H_*(X, \mathbb{Z}) \simeq \pi_*(X \otimes H\mathbb{Z}) \simeq 0$ for all $* < k$.
- The *Hurewicz homomorphism*
$$\mathbb{S} \otimes X \to H\mathbb{Z} \otimes X$$
induces an isomorphism in homotopy groups at π_k.

Proof. We apply 5.14, with $R = \mathbb{S}$. Hence, first proposition is clear and

$$\pi_k(H\mathbb{Z} \otimes X) \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \pi_k X \simeq \pi_k X$$

□

Corollary 5.16. Whitehead theorem. Let $f : X \to Y$ be a morphism of bounded below spectra. Then f is an equivalence iff it induces an integral equivalence, i.e.

$$H\mathbb{Z} \otimes X \to H\mathbb{Z} \otimes Y$$

is an equivalence.

Proof. Only if follows from functoriality of $H\mathbb{Z} \otimes -$. Suppose we have integral equivalence. Construct exact sequence

$$X \xrightarrow{f} Y \to \mathrm{cofib}(f)$$

By the long exact sequence $\mathrm{cofib}(f)$ is bounded below and is $H\mathbb{Z}$-acyclic. If $\mathrm{cofib}(f) \not\simeq 0$ it has a minimal nontrivial homotopy group, giving nontrivial homology group, contradicting Hurewicz, 5.15. □

Corollary 5.17. A map of spectrum is an $\mathbb{F}_p$ equivalence if it is a p-adic equivalence. Conversely, if both spectrum are bounded below, then $\mathbb{F}_p$ equivalence implies p-adic equivalence.

Proof. Step I. A standard fact? Note that if we have an SES

$$0 \to A \to B \to C \to 0$$

in abelian groups, the map in spectra

$$HA \to HB \to HC$$

is a cofiber sequence. This follows from the associated LES. Now by comparing the cofiber sequences

$$H\mathbb{Z} \to H\mathbb{Z} \to H\mathbb{F}_p$$

$$\mathbb{Z} \otimes \mathbb{S} \to \mathbb{Z} \otimes \mathbb{S} \to \mathbb{Z} \otimes \mathbb{S}/p$$

we obtain $\mathbb{Z} \otimes \mathbb{S}/p \simeq \mathbb{F}_p$.

Step II. The first proposition follows from applying $- \otimes \mathbb{Z}$. The second follows from Whitehead theorem 5.16 and preservation of connectendes. i.e. if one map below is an equivalence, so is the other

$$X \otimes \mathbb{S}/p \to Y \otimes Y/p$$

$$X \otimes \mathbb{F}_p \simeq X \otimes \mathbb{S}/p \otimes \mathbb{Z} \to Y \otimes \mathbb{S}/p \otimes \mathbb{Z} \simeq Y \otimes \mathbb{F}_p$$

□

6 Appendix B

6.1 Properties of Tate construction

Lemma 6.1. [NS18, I.2.6] Let Y be a spectrum with G-action for some finite group G/ The natural maps

$$Y^{hG} \to \varprojlim_n (\tau_{\le n} Y)^{hG}$$
$$Y_{hG} \to \varprojlim_n (\tau_{\le n} Y)_{hG}$$
$$Y^{tG} \to \varprojlim_n (\tau_{\le n} Y)^{tG}$$

are equivalences. We have a similar one for $\varinjlim$.

Proof. The first part follows as $(-)^{hG}$ commutes with limits. It suffices to prove the case $(-)_{hG}$. Being a colimit $(-)_{hG}$ is exact. As $\mathcal{C}_{\le n} \hookrightarrow \mathcal{C}$ is a colocalization, it is closed under colimits. The fiber of the map

$$\mathrm{fib}_n \to Y_{hG} \to (\tau_{\le n} Y)_{hG}$$

is thus n-connected. The functor $\varprojlim$ is exact, as it commutes with fiber sequences. Applying to the above fiber sequence, we obtain

$$\mathrm{fib} = \varprojlim_n \mathrm{fib}_n \to Y_{hG} \to \varprojlim_n (\tau_{\le n} Y)_{hG}$$

Completeness shows $\mathrm{fib} \simeq 0$: one may start with various definitions of completeness, whatever form, one consequence is $X \simeq \varprojlim_n \tau_{\le n} X$ for all X. Hence, we write each $\mathrm{fib}_n \simeq \varprojlim_k \tau_{\le k} \mathrm{fib}_n$. This yields

$$\varprojlim_n \varprojlim_k \tau_{\le k} \mathrm{fib}_n \simeq \varprojlim_k \varprojlim_n \tau_{\le k} \mathrm{fib}_n \simeq \varprojlim_k 0 \simeq 0$$

□

Lemma 6.2. Let $X \in Sp^G$. Smashing with Moore spectrum commutes with Tate construction.

Proof. $(-)^{tG}$ is exact, since both $(-)_{hG}, (-)^{hG}$ are exact. This then follows from definition of Moore spectrum. □

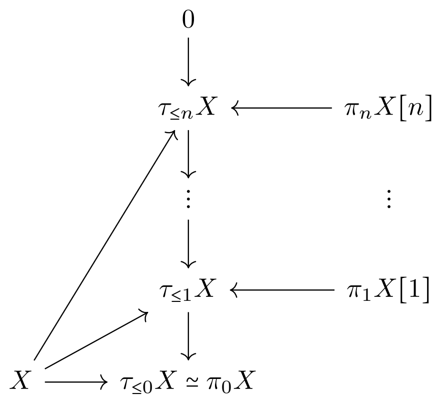
Lemma 6.3. [NS18, I.2.9] Let X be a spectrum with C_p action which is bounded below. Then X^{tC_p} is p-complete and equivalent to $(X_p\hat{})^{tC_p}$.

Proof. Step I. Reducing to the case when X is bounded. The canonical map $X \to X_p\hat{}$ is a p-adic equivalence, i.e. smashing with $\mathbb{S}/p$ is an equivalence. By 6.2,

$$X^{tC_p} \to (X_p\hat{})^{tC_p}$$

is a p-adic equivalence. By 1.15, $X_p\hat{}$ is also bounded below. Thus result follows if X^{tC_p} is p-complete. By 1.1, limits of p-complete spectra are p-complete. It suffices to prove the case for bounded X.

Step II. Apply the Postinkov tower and reduce to EM spectrum. Wlog, suppose $X \in \mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq 0}$. Decompose X by its Postnikov tower,



where $\pi_i X[i]$ are discrete spectrum which fit into fiber sequences

$$\pi_i X[i] \to \tau_{\leq i} X \to \tau_{\leq i-1} X$$

From structure, 1.1, p-complete spectra are closed under extensions. By inductions it suffices to prove the case for Eilenberg-MacLane spectrum

$$M[i]$$

Step III. Reduce to problem of classical Tate cohomology. By definition $\pi_i M[i]^{tC_p} \simeq \hat{H}^*(C_p, M)$, the Tate cohomology. This has p-torsion. Hence, the map

$$M[i] \xrightarrow{p} M[i]$$

is 0. Thus by characterisation 1, $M[i]$ is (p)-complete. □

Corollary 6.4. The statement holds if C_p is replaced any p-group[10] G.

Proof. The exact same proof follows, where it suffices to in *Step III* to show that the group cohomology of any module M over G is p-torsion. The nth cohomology groups are a module over $H^0(G, \mathbb{Z}) \simeq \mathbb{Z}/|G|$. □

6.2 Proper Tate construction revisted

As usual, all groups G here are considered as finite discrete groups.

Definition 6.5. Let $\mathcal{O}_G$ denote the *orbit category.* It is the full subcategory of $\mathcal{S}_G$ spanned by G/H.

Remark 6.6. This is equivalently the category of finite transitive G-sets with G-maps. Indeed, for any such set X, the orbit space of any chosen point $x \in X$ is equivalent to G/H_x, where H_x is the stabilzier of x.

We identify G-spaces as presheaves via Elmendorf's.

Theorem 6.7. Elemendorf. The functor

$$\mathrm{Fun}(G, \mathcal{S}) \to \mathrm{Fun}(\mathcal{O}_G^{op}.\mathcal{S})$$

sending $X \mapsto \mathrm{Map}_G(-, X)$ is a localization, inverting G-weak equivalences.

[10] Order of group is a power of p

Definition 6.8. In light of Thm. 6.7, given a G-space X, we denote $X(G/H)$ by X^H.

Definition 6.9. Let $\mathcal{F}$ be a family of subgroups of G. Then the *classifying space for a family of subgroups* is given by

$$E\mathcal{F}(G/H) \coloneqq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \varnothing & \text{if } H \notin \mathcal{F} \end{cases}$$

We will be using the special case $\mathcal{F} = \mathcal{P}$ the proper subgroups of G.

Corollary 6.10.

$$\mathrm{Map}_{\mathcal{S}_G}(X, E\mathcal{P}) \simeq \begin{cases} \varnothing & \text{if } X^G \not\simeq \varnothing \\ * & \text{if } X^G \simeq \varnothing \end{cases}$$

Definition 6.11. We let $\widetilde{E\mathcal{P}}$ denote the cofiber in pointed spaces,

$$(E\mathcal{P}_G)_+ \to S^0 \to \widetilde{E\mathcal{P}_G}$$

This is commonly referred to as the *isotropy sequence*. The first map is induced by the teminal map $E\mathcal{P} \to *$.

Definition 6.12. Let $\mathcal{O}_G^- \hookrightarrow \mathcal{O}_G$ be the the full subcategory of $\mathcal{O}_G$ spanned by G/H where $H \subsetneq G$. This is equivalently the transitive G with nontrivial G-action.

Lemma 6.13. $E\mathcal{P} \simeq \mathrm{colim}_{G/H \in \mathcal{O}_G^-} G/H$.

Proof. This is by the formula of Kan extension along Yoneda embedding and 6.10,

$$E\mathcal{P} \simeq \varinjlim_{(\mathcal{O}_G)_{/E\mathcal{P}}} G/H \simeq \varinjlim_{\mathcal{O}_G^-} G/H$$

□

Definition 6.14. Geometric fixed points.

$$\Phi^G : \mathrm{Sp}^G \to \mathrm{Sp}$$

is given by

$$X \mapsto (X \otimes \Sigma_G^\infty \widetilde{E\mathcal{P}_G})^G$$

Corollary 6.15. Geometric fixed points commutes with all colimits.

Proof. Note that by construction, categorical fixed points

$$(-)^G : \mathrm{Sp}^G \to \mathrm{Sp}$$

commutes with all colimits and limits. This is because it is given by a diagram in Pr_ω^R of compactly generated categories.[11] Hence it commutes with limits and filtered colimits. As Sp^G is stable, it commues with all colimits. As $\otimes$ commutes with all colimits separately, result follows. □

Corollary 6.16. Let $X \in \mathrm{Sp}^G$, $X^{\tau G}$ fits into a cofiber sequence

$$\mathrm{colim}_{\mathcal{E}}((X)^{hH})_{hWH} \to X^{hG} \to X^{\tau G}$$

for some contractible category $\mathcal{E}$.

[11] morphisms are right adjoints which commutes with filtered colimits.

Proof. Step I. First resduction.

$$\begin{aligned}
X^{\tau G} &\simeq \Phi^G(\beta X) \\
&\simeq \operatorname{cofib}\big((\Sigma^\infty_G (E\mathcal{P}_G)_+ \otimes \beta X)^G \to (\Sigma^\infty_G S^0 \otimes \beta X)\big) \\
&\simeq \operatorname{cofib}\big(\operatorname{colim}_{\mathcal{O}_G^-}(\Sigma^\infty_G (G/H)_+ \otimes \beta X)^G \to (\Sigma^\infty_G S^0 \otimes \beta X)^G\big) \\
&\simeq \operatorname{cofib}(\operatorname{colim}_{\mathcal{O}_G^-}(\beta X)^H \to \beta X)
\end{aligned}$$

Step II. Applying a projection to compute colimit We consider the functor

$$\mathcal{O}_G^- \to p\mathcal{O}_G^- = \{\text{cjg. class. of prop. subgroups of } G\}$$

$$H \mapsto \text{ cjg class of } H$$

If we let

$$\mathcal{T} : \mathcal{O}_G^- \to \mathrm{Sp}, E \mapsto E^H$$

be the functor, of interest, its lke, $p_!\mathcal{T}$, along this projection shows

$$\operatorname{colim} \mathcal{T} \simeq \operatorname{colim} p_! \mathcal{T} \simeq \operatorname{colim}_{G/H \in p\mathcal{O}_G^-} \operatorname{colim}_{(\mathcal{O}_G^-)_{/G/H}} \mathcal{T}$$

but note that each of the fiber is equvalent to BWH, where $WH := NH/H$ is the *Weyl group*.

Step III. finishing off the last equivalence.

$$\operatorname{colim}_{\mathcal{O}_G^-}(\beta X)^H \simeq \operatorname{colim}_{p\mathcal{O}^-{}_G}((\beta X)^H)_{hWH}$$

Now by [Nar18, Prop. 11], we have

$$\beta X^H \simeq X^{hH}$$

□

Now we finish by proving the little lemma required.

Lemma 6.17. Let $X \in \mathrm{Sp}^G$ then

$$(\Sigma^\infty_G G/H_+ \otimes \beta X)^G \simeq X^H$$

6.3 Pro-objects

We describe some basic notions underlying pro-objects.

Definition 6.18. Let $\mathcal{C}$ be any category with finite limits. $\mathrm{Pro}(\mathcal{C})$, is an infinite category with *all* limits staisfying

- There exists a limit preserving map $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$.
- Given any category $\mathcal{D}$ with all limits, restriction induces an equivalence

  $$\mathrm{Fun}^R(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{D})$$

  The lhs is the category of limit preserving functors, whilst the rhs is the category of finite limit preserving functors.

Remark 6.19. Though I have written $\mathcal{C} \hookrightarrow \text{Pro}(\mathcal{C})$, by construction, we can show $\mathcal{C}$ generates Pro(Ç) under cofiltered limits, and $\mathcal{C}$ includes fully faithfully.

Definition 6.20. An object in $\text{Pro}(\mathcal{C})$ is *constant* if it is equivalent to an object in the image of $\mathcal{C} \to \text{Pro}(\mathcal{C})$.

Corollary 6.21. If $\mathcal{C}$ have finite limits. A cofiltere diagram $F : I \to \mathcal{C}$ is constant iff

- F admits a limt in $\mathcal{C}$.
- For any finite limit preserving, the inverse limit of F is preserved by G

Our interest of application would to the following class of categories.

Definition 6.22. A *stable homotopy theory.* $(\mathcal{C}, \otimes, 1)$ is a presentable, symmetric monoidal stable category where the tensor product commutes with all colimits separately.

Example 6.23. Any filtered colimit commutes with finite limit in the category of abelian group. This follows from the fact the $U : \text{Ab} \to \text{Set}$ preserves and reflects filtered colimits. [12]

Corollary 6.24. [Mat16, 3.13] Let $(\mathcal{C}, \otimes, 1)$ be a stable homotopy theory. For any cofiltered diagram ,$F : I \to \mathcal{C}$, if the induced pro-object is constant, then for any $X \in \mathcal{C}$,

$$\varinjlim_I F(i) \otimes X \to \varprojlim_I F(i) \otimes X)$$

is an equivalence.

[12]There is a more general statement, for any cocomplete category which models an algebraic theory: these are generally reflective, flitered closed subcategory of a presheaf category. Then the result follows pointwise.

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