Exercise sheet 1

Exercise 1. Recall that the group completion functor $(-)^{\mathrm{gp}} \colon \mathrm{CMon} \to \mathrm{Ab}$ is left adjoint to the inclusion $\mathrm{Ab} \subset \mathrm{CMon}$.

(a) A commutative monoid M has the *cancellation property* if $a + c = b + c$ implies $a = b$. Show that the unit map $M \to M^{\mathrm{gp}}$ is injective if and only if M has the cancellation property.

(b) Suppose that $(M, +, \cdot)$ is a semiring (i.e., $(M, +)$ is a commutative monoid, $(M, \cdot)$ is a monoid, and the distributivity law holds). Show that the group completion M^{gp} (with respect to $+$) admits a unique ring structure such that $M \to M^{\mathrm{gp}}$ is a morphism of semirings.

(c) Show that the inclusion $\mathrm{Ab} \subset \mathrm{CMon}$ also has a right adjoint and describe it explicitly.

(d) Let X be a topological space. Show that $\mathrm{Maps}(X, \mathbb{N})^{\mathrm{gp}} \simeq \mathrm{Maps}(X, \mathbb{Z})$, where $\mathrm{Maps}(X, Y)$ denotes the set of continuous maps and $\mathbb{N}$ and $\mathbb{Z}$ are viewed as discrete topological spaces.

(e) Let Mon be the category of monoids and Grp the category of groups. Show that the inclusion $\mathrm{Grp} \subset \mathrm{Mon}$ has a left adjoint L and that if M is a commutative monoid then $L(M) \simeq M^{\mathrm{gp}}$.

Exercise 2.

(a) Let G be a finite group. Show that the functor associating to a finite G-set X the G-representation $\mathbb{C}[X]$ induces a morphism of rings $A(G) \to R(G)$ from the Burnside ring to the representation ring (use Exercise 1(b)).

(b) Let p be a prime number and C_p a cyclic group of order p. Show that there are isomorphisms of rings

$$A(C_p) \simeq \mathbb{Z}[x]/(x^2 - px) \quad \text{and} \quad R(C_p) \simeq \mathbb{Z}[y]/(y^p - 1).$$

What is the map $A(C_p) \to R(C_p)$ from (a) under these identifications?

Exercise 3. Let $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and let $P = \ker(\sigma)$ where

$$\sigma \colon A^3 \to A, \quad \sigma(f, g, h) = xf + yg + zh.$$

Show that there is an isomorphism of A-modules $P \oplus A \simeq A^3$ but that P is not free.

Hint. Let S^2 be the sphere of unit vectors in $\mathbb{R}^3$. Observe that every $p \in P$ induces a continuous function $S^2 \to \mathbb{R}^3$. Use this to show that if P were free, then the tangent bundle to S^2 would be trivial, which is false.

Exercise 4. Let R be a ring and $I \subset R$ a two-sided nilpotent ideal (i.e., $I^n = 0$ for some n). Extension of scalars defines a functor $\mathrm{Proj}(R) \to \mathrm{Proj}(R/I)$, $P \mapsto P/IP$. Show that:

(a) For every $P, Q \in \mathrm{Proj}(R)$, the induced map $\mathrm{Isom}_R(P, Q) \to \mathrm{Isom}_{R/I}(P/IP, Q/IQ)$ is surjective, where Isom_R denotes the set of R-linear isomorphisms.

(b) $\mathrm{Proj}(R) \to \mathrm{Proj}(R/I)$ induces a bijection between isomorphism classes of objects.

In particular, $K_0(R) \to K_0(R/I)$ is an isomorphism (we say that K_0 is *nilinvariant*).

Hint. For both (a) and (b), reduce first to the case $I^2 = 0$. Note that the injectivity in (b) follows from (a). To prove the surjectivity in (b), first show that every idempotent $\bar{e} \in R/I$ lifts to an idempotent in R. If e is any lift of $\bar{e}$, this amounts to solving the equation $(e + x)^2 = e + x$ in R; this equation becomes easier to solve by adding the constraint $ex = xe$.