Exercise sheet 8

Exercise 1. Let $\mathcal{C}$ be an exact category. Show that there is an equivalence of categories $Q\mathcal{C} \simeq Q(\mathcal{C}^{\mathrm{op}})$.

Exercise 2. Let X be a scheme.

(a) Show that the category $\mathrm{Vect}(X)$ of finite locally free $\mathcal{O}_X$-modules is an exact category if one takes admissible epimorphisms to be epimorphisms of sheaves.

(b) Show that a morphism $u\colon \mathcal{E} \to \mathcal{F}$ in $\mathrm{Vect}(X)$ is an admissible monomorphism if and only if, for every $x \in X$ with residue field $\kappa(x)$, the induced morphism of $\kappa(x)$-vector spaces $u(x)\colon \mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x) \to \mathcal{F} \otimes_{\mathcal{O}_X} \kappa(x)$ is injective.

Exercise 3. Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ a full subcategory containing 0 and closed under extensions. Show that $\mathcal{C}$ admits an exact structure in which a morphism is an admissible monomorphism (resp. an admissible epimorphism) if and only if it is a monomorphism (resp. an epimorphism) in $\mathcal{A}$ whose cokernel (resp. kernel) belongs to $\mathcal{C}$. Moreover, the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{A}$ is then exact.

Remark. Conversely, for any exact category $\mathcal{C}$, one can show that there exists an abelian category and an exact fully faithful embedding $\mathcal{C} \subset \mathcal{A}$, closed under extensions, such that the exact structure of $\mathcal{C}$ is inherited from $\mathcal{A}$. In fact, one can take $\mathcal{A}$ to be the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ that send short exact sequences to left exact sequences.

Exercise 4. A ring R is called *flasque* if there exists an R-bimodule M, finitely generated and projective as a left R-module, and an isomorphism of bimodules $\theta\colon R \oplus M \simeq M$. An example is the ring of A-linear endomorphisms of $\bigoplus_{\mathbb{N}} A$ for any ring A.

Let R be a flasque ring.

(a) Show that $K_0(R) = 0$.

(b) More generally, show that the space $K(R)$ is weakly contractible as follows. Construct a finite-product-preserving functor $\infty\colon \mathrm{Proj}(R) \to \mathrm{Proj}(R)$ with a natural equivalence $\infty(P) \simeq P \oplus \infty(P)$ (here, $\mathrm{Proj}(R)$ is the category of finitely generated projective *left* R-modules). Using the universal property of group completion, deduce that there exists a morphism $\infty\colon K(R) \to K(R)$ which is homotopic to $\mathrm{id}+\infty$, where $+\colon K(R)\times K(R) \to K(R)$ is induced by the functor $\oplus\colon \mathrm{Proj}(R)\times\mathrm{Proj}(R) \to \mathrm{Proj}(R)$. Using Eckmann–Hilton, deduce that $\pi_n K(R) = 0$ for all $n \geq 1$.

Remark. It follows from the group completion theorem that the space $BGL(R)$ is acyclic.