

Algebraic K-theory

$K_n : \text{Rings} \rightarrow \text{Ab}, n \in \mathbb{Z}$

$$K_0(R) = \mathbb{Z} \left[\begin{smallmatrix} \text{iso classes} \\ \text{of f.g. proj. } R\text{-modules} \end{smallmatrix} \right] / [M \oplus N] = [M] + [N].$$

Some history.

K_0 :	Grothendieck	1957	(also $K_{<0}$)	1957: Bott proved Bott periodicity
K_1 :	Bass	1964		$\pi_* \mathcal{U}$
K_2 :	Milnor	1967		1959: Atiyah-Hirzebruch
K_* :	Quillen	1971		$K^*(X)$ for X a top. space

In fact, Quillen defined a space $K(R)$, $K_*(R) = \pi_{*R} K(R)$.

$$(* \geq 0)$$

Fundamental computations:

Quillen 1971: $K_*(F_q)$

Borel 1974: $K_*(G_F) \otimes \mathbb{Q}$ for F a number field

Soulé 1984: $K_*(k, \mathbb{Z}/n)$ for $k = \bar{k}$, $n \in k^\times$

Some motivation.

Number theory

1) Relation to ζ -functions. $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$ $s \in \mathbb{C}$

X scheme of finite type over \mathbb{Z}

$$\zeta_X(s) = \prod_{\substack{x \in X \\ \text{closed pts}}} \frac{1}{1 - |K(x)|^{-s}} \quad \text{converges for } \operatorname{Re}(s) > \dim(X).$$

Conjecture (Soulé) X regular

$$\sum_{n \in \mathbb{Z}} \operatorname{ord}_{s=n} \zeta_X(s) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}} (K_i(X) \otimes \mathbb{Q})_{\text{inv}}$$

ℓ eigenspace
of Adams operator

Conjecture (Lichtenbaum) $X = \operatorname{Spec} B_F$, $(F:\mathbb{Q}) < \infty$. \mathfrak{U}^n .

$$\sum_{n \geq 0} \operatorname{sp. val.} \zeta_X(-n) = \frac{\# K_{2n}(B_F)}{\# K_{2n+1}(B_F)_{\text{tors}}} \cdot (\text{n-th Borel regulator of } F)$$

up to $\pm 2^n$

Soulé's conj. is known when $n \geq \dim(X)$

For $n = \dim(X) - 1 \Rightarrow$ Birch-Swinnerton-Dyer conjecture

Lichtenbaum's conj: known for $n=0$ (class number formula)
known for F/\mathbb{Q} abelian

2) The Kummer-Vandiver conjecture: $\# \text{Pic}(G_F)$

Conj: If p is prime, p does not divide the class number of $\mathbb{Q}(\zeta_p) \cap \mathbb{R}$. (known for $p < 163\,000\,000$).

Then (Kurihara) Kummer-Vandiver $\Leftrightarrow K_{4n}(\mathbb{Z}) = 0$ for $n > 0$.

- $K_i(\mathbb{Z})$ are known for $i \not\equiv 0 \pmod{4}$.
- $K_4(\mathbb{Z}) = 0$ (Reynes 2000)
- $K_8(\mathbb{Z}) = 0$ (2019) Kupers
- $K_{12}(\mathbb{Z}) = ?$

Geometric topology (inc. rings)

3) Wall finiteness obstructions.

Q: Let Y be a finite CW complex and $X \subset Y$ a retract (i.e. $\exists p: Y \rightarrow X$ s.t. $p|_X = \text{id}_X$)

Is X homotopy equivalent to a finite CW complex?

A: Not always. Wall defined an obstruction in $K_0(\mathbb{Z}[\pi_1 X])$

$\tilde{X} \rightarrow X$ universal cover $\pi_1 X \hookrightarrow \tilde{X}$

$C_*(\tilde{X})$ is a chain complex of $\mathbb{Z}[\pi_1 X]$ -modules.

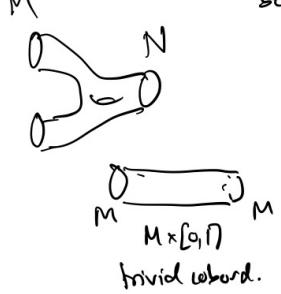
$$\chi(C_*(\tilde{X})) := \sum_i (-1)^i [C_i(\tilde{X})] \text{ in } K_0(\mathbb{Z}[\pi_1 X]).$$

Then (Wall) X is htpy equiv to a finite CW complex

$$\Leftrightarrow \chi(C_*(\tilde{X})) = 0 \text{ in } K_0(\mathbb{Z}[\pi_1 X]).$$

4) Whitehead torsion:

M, N smooth compact manifolds of dim n . A cobordism between M and N is a $(n+1)$ -dim manifold W s.t. $\partial W = M \sqcup N$



Q: Is every h-cobordism trivial?

A: No. There is an obstruction in $K_1(\mathbb{Z}[\pi_1 M])$

Then (s -cobordism theorem) $\text{dim}(M) \geq 5$

$$\left\{ \begin{array}{l} \text{diffeo classes of} \\ \text{h-cobordisms } M \sim M \end{array} \right\} \simeq K_1(\mathbb{Z}[\pi_1 M]) / \pm \pi_1 M =: Wh(M)$$

- Rmk:
- $n=4$: this is false
 - $n=3$: open question (\Leftrightarrow existence of exotic smooth structures on S^4)
 - $n=2$: \Leftrightarrow Poincaré conjecture (theorem of Perelman) on S^4
 - Wildenhain generalized 3) & 4) to something involving $K_*(\mathbb{Z}[H_1X])$.

Algebraic geometry

- 5) Relation with algebraic cycles
 X scheme.

$$Z(X) := \mathbb{Z}[\text{irr. subvarieties of } X]$$

(e.g. $X = \text{Spec } R$, $Z(X) = \mathbb{Z}[\text{prime ideals in } R]$).

Intersection theory: $Z(X) \times Z(X) \rightarrow Z(X)$?
 $(Y_1, Y_2) \mapsto Y_1 \cap Y_2$.

This works up to rational equivalence (at least for X smooth over a field)

Chow ring: $A(X) = Z(X)/\sim_{\text{rat}}$

(algebraic analogue of singular cohomology)

Theorem (Grothendieck) $A(X) \otimes \mathbb{Q} \cong K_0(X \otimes \mathbb{Q})$ as rings.

Chern class
 $\text{ch}: K_0(X) \xrightarrow{\text{ch}} A(X) \otimes \mathbb{Q}$.

- 6) Riemann-Roch theorem. (1865)

X Riemann surface: compute the dimension of the space of meromorphic functions on X with prescribed zeros and poles
 \equiv sections of some line bundles

Grothendieck 1957:

- \mathbb{C} no arbitrary field
- curves no arbitrary smooth variety
- line bundle no vector bundle
- relative version: for $f: X \rightarrow Y$ proper morphism

$$K_0(X) \xrightarrow{\text{ch}} A(X) \otimes \mathbb{Q}$$

$$f_* \downarrow \quad \quad \quad \downarrow f_* \quad \text{commutes up to } \text{td}(T_f)$$

$$K_0(Y) \xrightarrow{\text{ch}} A(Y) \otimes \mathbb{Q} \quad \quad \quad A(X) \otimes \mathbb{Q}$$

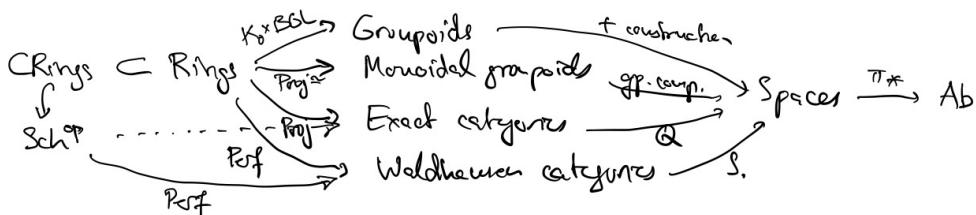
Rmk Bloch 1986 extended 5)&6) to higher K-theory / higher Chow groups.

Some conjectures about algebraic K-theory:

- Bass conjecture: If X regular scheme of f.t. over \mathbb{Z} , $K_i(X)$ is finitely generated.
 \Downarrow
 (known if $\dim(X) \leq 1$ by Quillen)
- Beilinson-Soulé vanishing conjecture: k field $\Rightarrow (K_{2g-p}(k) \otimes \mathbb{Q})_{(1)} = 0$
 (known for $k = \mathbb{F}_q$, number fields) for $p < 0, q \geq 0$.
- Wübel vanishing conjecture: If X noetherian of $\dim d$, then $K_{\geq i}(X) = 0$
 (fully proved in 2016 by Kerz-Strunk-Tamme) for $i > d$.
- Quillen-Lichtenbaum conjecture: relates K-theory to étale cohomology
 (proved by Voevodsky 2010)
- Redshift conjecture (Rognes): relates K-theory to chromatic homotopy theory

Constructions of K-Theory

- Plus construction (Quillen)
- Group completion (Segal)
- \mathbb{Q} -construction (Friedlander)
- S_+ -construction (Waldhausen).



$\text{Proj}(R) = \text{category of f.g. projective } R\text{-modules}$

$\text{Perf}(R) = \text{perfect complexes of } R\text{-modules}$