

Classifying spaces

Recall: G group \rightsquigarrow groupoid BG

Def The classifying space of G is $|N(BG)| \in \text{Top}$

Remark The notation "BG" is used for:
 - the groupoid BG
 - $BG = |N(BG)| \in \text{Top}$
 - $BG = N(BG) \in \text{sset}$

(*) Proposition Let C be a groupoid, $x \in C$. Then

$$\pi_0(C) \cong \pi_0(|N(C)|)$$

$$\pi_1(C, x) = \pi_1(|N(C)|, x)$$

$$\pi_n(|N(C)|, x) = 0 \text{ for } n \geq 2.$$

(In other words, $|N(C)|$ is 1-truncated)

Remark $C, D \in \text{Cat}$

$\xrightarrow{\text{N}(-)}$

$|N(C)| \times \Delta^1 \longrightarrow |N(D)|$

homotopy from $N(f)$ to $N(g)$.

$\xrightarrow{\text{!-!}}$

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Hence, if $f: C \rightarrow D$ is an equivalence of categories (or has an adjoint)
 then $N(f)$ and $|N(f)|$ are homotopy equivalences

Proof of Prop (*)

Choose $\coprod_i BG_i \rightarrow C$ equivalence $\Rightarrow |N(C)| = \coprod_i |N(BG_i)|$

homotopy equivalence

\Rightarrow wlog $C = BG$

Consider the groupoid EG with:
 - objects are $g \in G$
 - morphisms are $g \xrightarrow{h} gh$

$EG \rightarrow BG$	$\bullet EG \cong *$
$\begin{matrix} g & \mapsto & * \\ (g \xrightarrow{h} gh) & \mapsto & h \end{matrix}$	\bullet The group G acts on EG by multiplication from the left. $N(EG) \quad G \sqsubseteq G \times G \sqsubseteq G \times G \times G \dots$ G acts on the first factor $\downarrow \quad \downarrow \quad \downarrow \pi_1 \quad \downarrow \pi_{23}$ $N(BG) \quad * \sqsubseteq G \sqsubseteq G \times G \sqsubseteq \dots$

$\Rightarrow G$ acts freely on $|N(EG)|$ and $|N(EG)|/G \simeq |N(BG)|$

$\Rightarrow |N(EG)| \rightarrow |N(BG)|$ is a covering map with group G and $|N(EG)| \simeq *$ (contractible)

By covering space theory, $\pi_1(|N(BG)|) \simeq G$

$\pi_n(|N(BG)|) = 0$ for $n \geq 2$. \square

Examples: • $B\mathbb{Z} \simeq S^1$ (homotopy equivalence)

$$\begin{bmatrix} G & \xrightarrow{\quad R \quad} & S^1 \\ \mathbb{Z} & & \end{bmatrix}$$

$$\bullet BC_2 \simeq RP^\infty \quad \begin{bmatrix} S^\infty & \xrightarrow{\quad R \quad} & RP^\infty \\ C_2 & & \end{bmatrix}.$$

• BC_n "infinite lens spaces"

Remark. The previous proposition fails for categories. Thomason showed that every space is weakly equivalent to $|N(C)|$ for some category C .

In particular, $|N(C)|$ is not 1-truncated in general.

CW complexes

Definition A CW structure on a top. space X is:

1) A sequence of subspaces $X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \dots \subset X^{(n)} \subset \dots \subset X$ such that $X = \text{colim}_n X^{(n)}$ and $X^{(n)}$ is discrete.

2) For $n \geq 1$, pushout squares

$$\begin{array}{ccc} \coprod_{\alpha \in I_n} S^{n-1} & \xrightarrow{\partial_\alpha} & X^{(n-1)} \\ \downarrow & \text{PO} & \downarrow \\ \coprod_{\alpha \in I_n} D^n & \xrightarrow{e_\alpha} & X^{(n)} \end{array} \quad \begin{array}{l} \text{"}X^{(n)} \text{ is obtained from } X^{(n-1)} \\ \text{by attaching } n\text{-cells"} \end{array}$$

The maps $e_\alpha: D^n \rightarrow X^{(n)}$ are the n -cells and $\partial_\alpha: S^{n-1} \rightarrow X^{(n-1)}$ are the attaching maps.

A CW complex is a top-space that admits a CW structure.
 $\begin{cases} \text{finite-dimensional: } X = X^{(n)} \text{ for some } n \\ \text{finite: } \bigcup_{n \geq 0} I_n \text{ is finite} (\Leftrightarrow \text{compact}) \end{cases}$

Notation $\text{CW} \subset \text{Top}$ full subcategory of CW complexes

Remark: $\text{CW} \Rightarrow$ Hausdorff, paracompact, compactly generated, ...

Example: 1) Every smooth manifold is a finite-dim CW complex.

2) If X is a simplicial set, $|X| \cong$ a CW complex:

$X = \bigcup_n sk_n(X)$ where $sk_n(X) \subset X$ simp. subset generated by non-degenerate k -simplices for $k \leq n$.

$$\begin{array}{ccc} \coprod \partial \Delta^n & \longrightarrow & sk_{n-1}(X) \\ \downarrow & p_0 & \downarrow \\ \coprod_{\substack{\text{non-deg} \\ \text{v-simplices} \\ \text{of } X}} \Delta^n & \longrightarrow & sk_n(X) \end{array}$$

Theorem.

- 1) For every $X \in \text{Top}$, there exists a CW complex X' and a weak equivalence $X' \xrightarrow{\sim} X$.
- 2) (Whitehead) Every weak equivalence between CW complexes is a homotopy equivalence.

Dipremion (Localization of categories)

If C is a category and W a collection of morphisms in C , there exists a functor $L: C \rightarrow C[W^{-1}]$ satisfying the following universal property:

- L sends W to isos
- for every category D , the functor $L^*: \text{Fun}(C[W^{-1}], D) \rightarrow \text{Fun}(C, D)$ is fully faithful and $F: C \rightarrow D$ is in the essential image iff $F(W) \subset \{\text{isos}\}$.

Example • $C \xrightarrow[L]{\text{full}} D \Rightarrow L$ exhibits D as $C[W^{-1}]$, where $W = L^{-1}(\text{isos})$.
"Bousfield localization"

• $H_n: \text{Top} \rightarrow \text{Ab}$ sends weak equivalences to isos.

$$\begin{array}{ccc} & \downarrow & H_n \\ \text{Top} & \xrightarrow{\text{full}} & \text{Ab} \\ & \text{Top}[W^{-1}] & \end{array}$$

$$\begin{array}{ccccc} & \xrightarrow{C_*^{\text{sing}}} & \text{Ch}_{\geq 0} & \xrightarrow{H_n} & \text{Ab} \\ \text{Top} & \downarrow & \downarrow & \swarrow & H_n \\ \text{Top}[W^{-1}] & \dashrightarrow & \text{Ch}_{\geq 0}[q, i] & \end{array}$$

Theorem (Quillen)

The functors $\mathrm{I} \cdot \mathrm{I} : \mathrm{sSet} \rightarrow \mathrm{Top}$ and $\mathrm{Sing} : \mathrm{Top} \rightarrow \mathrm{sSet}$ preserve weak equivalences and the induced adjunction

$$\mathrm{sSet}[w.e] \xrightleftharpoons[\mathrm{Sing}]{} \mathrm{Top}[w.e].$$

is an equivalence of categories (i.e.: unit $K \rightarrow \mathrm{Sing}(K)$ and counit $\mathrm{Sing}(X) \rightarrow X$ are w.e.)

Moreover: • if $X \in \mathrm{CW}$ and $Y \in \mathrm{Top}$, $\mathrm{Hom}_{\mathrm{Top}[w.e]}(X, Y) = \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\mathrm{htpy}$

• if $X \in \mathrm{sSet}$ and $Y \in \mathrm{Kan}$, $\mathrm{Hom}_{\mathrm{sSet}[w.e]}(X, Y) = \mathrm{Hom}_{\mathrm{sSet}}(X, Y)/\mathrm{htpy}$.

$$\begin{array}{ccc} \mathrm{sSet} & \xrightarrow{\mathrm{I} \cdot \mathrm{I}} & \mathrm{CW} \\ \cup & & \cap \\ \mathrm{Kan} & \xrightarrow{\mathrm{Sing}} & \mathrm{Top} \end{array}$$

Remark.

$\{(C, W), C \in \mathrm{Cat}, W \text{ collection of morphisms}\}$

$$(C, W) \longmapsto C[W].$$

$$\begin{array}{ccc} & \longrightarrow & \mathrm{Cat} \\ \downarrow & & \nearrow \\ \mathrm{Cat}_{\infty} & & \text{homotopy category.} \end{array}$$

"categories enriched in
 ∞ -groupoids"