

Definition • An E_1 -space (or A_∞ -space) is a functor $X: \Delta^{\text{op}} \rightarrow \text{Top/Set}$ s.t. the Segal maps induce weak equivalences

$$X_n \xrightarrow{(f_i)} X_1^n$$

(also called Γ -space)

- An E_∞ -space is a functor $X: \text{Fin}_* \rightarrow \text{Top/Set}$ s.t. $X(I_+) \xrightarrow{(f_i)} X(*_+)^I$ is a weak equivalence for all $I \in \text{Fin}_*$.
- The underlying space is X_1 resp. $X(*_+)$.
- A morphism of E_1/E_∞ -spaces is simply a natural transformation.
- If X, Y are E_1/E_∞ -spaces, a morphism $X \rightarrow Y$ is called a weak equivalence if every component is a weak equivalence (because of the Segal condition, it suffices to require that $X_1 \rightarrow Y_1$ is a w.e.).

Remarks 1) An E_∞ -space X has an underlying E_1 -space :

$$\Delta^{\text{op}} \rightarrow \text{Fin}_* \xrightarrow{X} \text{Top/Set}.$$

2) Let X be an E_1 -space.

$$X_2 \xrightarrow[\sim]{(d_0, d_1)} X_1 \times X_1$$

$$\begin{matrix} & d_1 \\ & \downarrow \\ X_1 & \end{matrix}$$

If $X_{1,2}$ are CW complexes or Kan complexes, then we can choose a homotopy move to $(d_0, d_1): X_2 \rightarrow X_1 \times X_1$ to get a multiplication $X_1 \times X_1 \rightarrow X_1$, well-defined up to homotopy.

In particular, X_1 has a monoid structure in $\text{Top}(\text{w.e.})$ and $\pi_0(X_1)$ is a monoid.

Example If $X \in \text{Top}$, $x \in X$, then $\Omega_x X$ has a structure of E_1 -space.

"Moore loops" Let $\Omega_x^M X = \{ (t, g) \mid t \geq 0, g: [0, t] \rightarrow X, g(0) = g(t) = x \}$

Then $\Omega_x^M X$ is a monoid in Top : $(t, g) \cdot (s, h) = (t+s, g \circ h)$

and the maps $\Omega_x X \hookrightarrow \Omega_x^M X \longrightarrow \Omega_x X$

$$g \mapsto (1, g)$$

$$(t, g) \mapsto g \circ ([0, t] \xrightarrow{t-s} [0, t])$$

are homotopy inverse to one another.

$$\rightsquigarrow E_i\text{-space} \quad * \in \Omega^M X \subseteq \Omega^M X \times \Omega^M X \quad \dots$$

Definition. • An E_i -space X is called an E_i -group or group-like if the monoid $\pi_0(X)$ is a group.

• Similarly for E_∞ -spaces. An E_∞ -group is also called a Picard ∞ -groupoid.

Example $\Omega_x X$ is an E_i -group since $\pi_0(\Omega_x X) = \pi_1(X, x)$ is a group.

Group completion: There is an adjunction

$$\text{Fun}_{\text{sgdg}}(\Delta^{\text{op}}, \text{Top}/\text{sSet})[\text{w.e.}] \xrightleftharpoons[\text{full}]{(-)^{\text{gp}}} \{E_i\text{-groups}\}$$

and similarly for E_∞ -spaces. Moreover, the following square commutes:

$$\begin{array}{ccc} E_i\text{-spaces} & \xrightarrow{(-)^{\text{gp}}} & E_i\text{-groups} \\ \text{forget} \uparrow & & \uparrow \text{forget} \\ E_\infty\text{-spaces} & \xrightarrow{(-)^{\text{gp}}} & E_\infty\text{-groups} \end{array}$$

Theorem (delooping theory)

1) There is an adjunction

$$\text{Fun}_{\text{sgdg}}(\Delta^{\text{op}}, \text{Top})[\text{w.e.}] \xrightleftharpoons[\Sigma_2]{B} \text{Top}_*[\text{w.e.}]$$

where B is the following composition

$$\begin{array}{c} \text{Fun}(\Delta^{\text{op}}, \text{Top}) \xrightarrow{\text{sing}} \text{Fun}(\Delta^{\text{op}}, \text{sSet}) \simeq \text{Fun}(\Delta^{\text{op}} \times \Delta^{\text{op}}, \text{Set}) \\ \downarrow \text{diag}^* \qquad \qquad \qquad \text{diag}: \Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}} \\ \text{Top} \xleftarrow{\text{!-!}} \text{sSet} \simeq \text{Fun}(\Delta^{\text{op}}, \text{Set}) \end{array}$$

2) This adjunction restricts to an equivalence between the full subcategories of E_i -groups and pointed connected spaces.

Remark: 1) this generalizes the equivalence of 2-categories

$$\begin{array}{ccc} \text{Groups} & \simeq & \text{pointed connected groupoids} \\ G & \longmapsto & BG \\ \text{Aut}(*) & \hookleftarrow & (X, *) \end{array}$$

2) If M is a monoid viewed as a discrete E_i -space,

the BM is the weak classifying space $|N(\bullet^M)|$.

Corollary If X is an E_i -space, then $X^{op} \simeq S^2 BX$

Remark If X is a discrete E_∞ -space, one can show that X^{op} is also discrete.

However, if X is a discrete E_1 -space (i.e. an ordinary monoid)

then X^{op} is not usually discrete.

(In fact, every E_1 -group is \simeq to X^{op} for some discrete X).

Def. Let R be a ring. The K-theory space $K(R)$ of R is

the group completion of the E_∞ -space $N(\text{Proj}(R)^\simeq)$

For $n \geq 0$, $K_n(R) := \pi_n(K(R), 0)$.

Remark The 1-truncation (= fundamental groupoid) of $K(R)$ is the groupoid $T_{\leq 1} K(R)$ previously defined,
by comparison of universal properties.

$$\left(\text{Gpd} \xrightleftharpoons[N]{\pi_1} \text{sSet} . \right)$$

Thm (Barrett–Priddy–Quillen theorem)

$$\pi_n(N(F_n)^\simeq) \simeq \pi_n^s(S^0) := \varprojlim_{k \rightarrow \infty} \pi_{n+k}(S^k) \quad \begin{cases} \text{known for} \\ n \leq \approx 30 \end{cases}$$

e.g. the Hopf map $S^3 \rightarrow S^2$ defines an element in $\pi_1^s(S^0)$

this corresponds to the non-zero element in $\pi_1(N(F_n)^\simeq) = \mathbb{Z}_2$.