

Quillen's plus construction

Let $p: E \rightarrow B$ be a continuous map with homotopy fibers F_b for $b \in B$

If $E \cong B \times F$ then $C_*(E) \cong C_*(B) \otimes C_*(F)$ (Künneth formula)

In general, we have the Serre spectral sequence:

$$H_*(B, \underbrace{H_*(F)}_{\substack{\text{local system on} \\ B \text{ given by } b \mapsto H_*(F_b)}) \Rightarrow H_*(E)$$

Using the Serre SS, one can prove the following:

Prop A continuous map $p: E \rightarrow B$ induces an isomorphism on homology with arbitrary local coefficient iff $\tilde{H}_*(F_b) = 0 \forall b \in B$.

Def. A map $p: E \rightarrow B$ is acyclic if it satisfies these conditions.

Warning: If $p: E \rightarrow B$ induces an iso on homology with constant coeff
 $\Leftrightarrow \tilde{H}_*(F_b) = 0$.

Examples

- $X \rightarrow *$ is acyclic $\Leftrightarrow \tilde{H}_*(X) = 0$ (we say X is acyclic)
- The Poincaré sphere:

$$\left(\text{solid dodecahedron, } \begin{array}{l} \text{glue opposing faces} \\ \text{with minimal clockwise rotation} \end{array} \right)$$

Removing the interior of the dodecahedron makes acyclic space

It is not contractible. In fact, π_1 is an extension of

A_5 by C_2 .

Proposition Let $p: E \rightarrow B$ be acyclic and $e \in E$.

- 1) $p_*: \pi_1(E, e) \rightarrow \pi_1(B, p(e))$ is surjective with perfect kernel.
- 2) If $\pi_1(E, e)$ has no nontrivial perfect subgroup for all $e \in E$, then p is a weak equivalence.

Pf. Exercise.

Definition A group is called hypabelian if it has no nontrivial perfect subgroup.

(Recall: any group has a maximal perfect subgroup.)

Theorem (Quillen)

Let X be a CW complex. There exist a CW complex X^+ and a map $\eta: X \rightarrow X^+$ such that:

1) η is acyclic and $\eta_*: \pi_1(X, x) \rightarrow \pi_1(X^+, \eta(x))$ kills the maximal perfect subgroup for every $x \in X$.

2) Let $f: X \rightarrow Y$ be a map such that $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(y))$ kills the maximal perfect subgroup for every $x \in X$. Then

there exists

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta \downarrow & \exists & \uparrow \\ X^+ & & \end{array}$$

unique up to homotopy.

Corollary Let $\text{Top}^{\text{hyp}} \subset \text{Top}$ be the full subcategory of spaces with hypabelian fundamental groups. Then there is an adjunction

$$\text{Top}^{[w, \bar{e}]} \rightleftarrows \text{Top}^{\text{hyp}} \text{[} w, \bar{e} \text{]} .$$

Group completion theorem (McDuff-Segal)

Let X be an E_∞ -space (more generally, a homotopy commutative E_1 -space).

Suppose there is a cofinal embedding $N \subset \pi_0(X)$. Let

$$X_\infty = \text{colim } (X \xrightarrow{\begin{smallmatrix} +1 \\ \uparrow \end{smallmatrix}} X \xrightarrow{\begin{smallmatrix} +1 \\ \uparrow \end{smallmatrix}} X \xrightarrow{\begin{smallmatrix} +1 \\ \uparrow \end{smallmatrix}} \dots)$$

$$\text{Then } X^\text{op} \cong X_\infty^+.$$

$$\mu(1, -): X \rightarrow X$$

$$\begin{array}{c} X \times X \xrightarrow{\text{susp}} X \times X \\ \downarrow \quad \uparrow \quad \text{adj} \\ \text{Top}^{[w, \bar{e}]} \end{array}$$

Corollary Let R be a ring. Then $K(R) \cong K_0(R) \times \text{BGL}(R)^+$.

(We'll come back to this later.)

Construction of X^+

WLOG: X is connected. Choose $x \in X^{(0)}$. Let $\mathbb{P} \subset \pi_1(X, x)$ maximal perfect subgroup, let $G = \pi_1(X, x)/\mathbb{P}$.

Step 1 Attach 2-cells to kill \mathbb{P} .

$\forall [\alpha] \in \mathbb{P}$, choose a representative $\alpha: S^1 \longrightarrow X^{(1)}$. Define X' by:

$$\begin{array}{ccc} \bigvee_{[\alpha] \in \mathbb{P}} S^1 & \longrightarrow & X \\ \downarrow & \text{PD} & \downarrow \pi_1 \\ \bigvee_{G \cap \mathbb{P}} D^2 & \xrightarrow{e_\alpha^2} & X' \end{array} \quad \begin{array}{ccc} \mathbb{Z}^{*\mathbb{P}} & \longrightarrow & \pi_1(X, x) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(X', x) \end{array}$$

By van Kampen theorem, this is a pushout square of groups

$$\Rightarrow \pi_1(X', x) = \pi_1(X, x)/\mathbb{P} = G$$

But: $X \rightarrow X'$ is not an iso on homology. It is an iso on H_n for $n \geq 4$, but not $n=2, 3$.

Step 2 Attach 3-cells to fix the homology.

Let $\tilde{X}' \rightarrow X'$ be the universal cover of (X', x)

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X}' \\ \downarrow \text{PB} & & \downarrow \\ X & \hookrightarrow & X' \end{array} \quad \Rightarrow \quad \tilde{X} \rightarrow X \text{ is a covering space with } \pi_1(\tilde{X}) = \mathbb{P} \subset \pi_1(X) \\ (\text{from the LES: } 0 \rightarrow \pi_1(\tilde{X}) \hookrightarrow \pi_1(X) \rightarrow \underbrace{\pi_0(\text{hofib})}_{\pi_1(X')} \rightarrow 0)$$

Let $\tilde{e}_\alpha^2: D^2 \rightarrow \tilde{X}'$ be a lift of $e_\alpha^2: D^2 \rightarrow X'$.

Then \tilde{X}' is obtained from \tilde{X} by attaching the 2-cells $g\tilde{e}_\alpha^2$, $g \in \pi_1(X') = G$.

\rightsquigarrow SES of chain complexes:

$$0 \rightarrow C_*^{\text{cell}}(\tilde{X}) \longrightarrow C_*^{\text{cell}}(\tilde{X}') \longrightarrow \bigoplus_{G \times \mathbb{P}} \mathbb{Z}[2] \rightarrow 0 \quad \text{P perfect}$$

\rightsquigarrow LES

$$H_2(\tilde{X}) \longrightarrow H_2(\tilde{X}') \longrightarrow \bigoplus_{G \times \mathbb{P}} \mathbb{Z} \xrightarrow{\partial} H_1(\tilde{X}) = \pi_1(\tilde{X})^{ab} = \mathbb{P}^{ab} = 0$$

$\simeq \uparrow$ (Hurewicz) $\nearrow \pi_2(\tilde{X}')$

Choose maps $S^2 \xrightarrow{\partial \tilde{e}_\alpha^3} \tilde{X}'^{(2)}$ lifting the 2-cells $[\tilde{e}_\alpha^2] \in \bigoplus_{G \times P} \mathbb{Z}$

Define X^+ by the pushout square:

$$\begin{array}{ccc} \coprod_{G \times P} S^2 & \xrightarrow{\partial \tilde{e}_\alpha^3} & \tilde{X}' \longrightarrow X' \\ \downarrow & & \downarrow \\ \coprod_{G \times P} D^3 & \longrightarrow & X^+ \end{array}$$

$$\text{van Kampen} \Rightarrow \pi_1(X^+) = \pi_1(X') = \pi_1(X)/P = G.$$

It remains to show that $X \hookrightarrow X^+$ is acyclic, i.e., induces an isomorphism on homology with any local coefficients.

Let $\tilde{X}' \rightarrow X'$ universal cover of (X', x) . As before,

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X}' \hookrightarrow \tilde{X} \\ \downarrow \pi & & \downarrow \text{univ} & & \tilde{X}' \text{ is obtained from } \tilde{X}' \text{ by attaching the} \\ X & \hookrightarrow & X' \hookrightarrow X^+ & & 3\text{-cells } g\tilde{e}_\alpha^3 \text{ for } g \in G = \pi_1(X') \text{ and } [\alpha] \in P. \end{array}$$

Look at $C_*^{\text{cell}}(\tilde{X}) \subset C_*^{\text{cell}}(\tilde{X}')$:

$$\begin{array}{ccccccc} \bigoplus_{I_4} \mathbb{Z} & = & \bigoplus_{I_4} \mathbb{Z} & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{I_3} \mathbb{Z} & \hookrightarrow & \bigoplus_{I_3 \cup (G \times P)} \mathbb{Z} & \twoheadrightarrow & \bigoplus_{G \times P} \mathbb{Z} & & \\ \downarrow & & \downarrow & & \downarrow d & & \\ \bigoplus_{I_2} \mathbb{Z} & \hookrightarrow & \bigoplus_{I_2 \cup (G \times P)} \mathbb{Z} & \twoheadrightarrow & \bigoplus_{G \times P} \mathbb{Z} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{I_1} \mathbb{Z} & = & \bigoplus_{I_1} \mathbb{Z} & \longrightarrow & 0 & & \end{array}$$

Claim: d is an isomorphism.

$$\text{In fact: } d(g\tilde{e}_\alpha^3) = g\tilde{e}_\beta^3 \quad \begin{matrix} \parallel \\ g \cdot d(\tilde{e}_\alpha^3) \end{matrix}$$

$$\begin{array}{c} S^2 \xrightarrow{\partial \tilde{e}_\alpha^3} \tilde{X}'^{(2)} \xrightarrow{\tilde{e}_\beta^2 / \partial \tilde{e}_\beta^3} \tilde{e}_\beta^2 \\ \searrow \quad \nearrow \\ S^2 \end{array}$$

degree is $\begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta. \end{cases}$

by construction.

$$\Rightarrow C_*^{\text{cell}}(\tilde{X}) \rightarrow C_*^{\text{cell}}(\tilde{X}') \text{ is quasi-iso}$$

$$\Leftrightarrow X \rightarrow X^+ \text{ is acyclic.}$$

