

K-Theory of exact categories (after Quillen I)

Definition

- 1) A category is pointed if it has an object $\mathbf{0}$ which is both initial and final.
- 2) A category is semi-additive (or pre-additive) if it is pointed and for every objects X, Y , the map $X \amalg Y \xrightarrow{\begin{pmatrix} \text{id}_X & 0 \\ 0 & \text{id}_Y \end{pmatrix}} X \times Y$ is an isomorphism.

$\Rightarrow \text{Hom}(X, Y)$ has a canonical structure of commutative monoid:

$$f, g \in \text{Hom}(X, Y)$$

$f+g$ is:

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$$

$$\downarrow \simeq \quad \downarrow \simeq$$

$$X \amalg X \xrightarrow{f \amalg g} Y \amalg Y \xrightarrow{\nabla} Y.$$

- 3) A category is additive if it is semi-additive and $\text{Hom}(X, Y)$ is a group for all X, Y .

Definition An exact category is an additive category \mathcal{C} with two collections of morphisms called "admissible monomorphisms" and "admissible epimorphisms", such that:

0) these two collections are closed under isomorphism in $\text{Fun}(\mathcal{E}\mathcal{T}, \mathcal{C})$.

1) for every $X \in \mathcal{C}$, $0 \rightarrow X$ is adm. mono
 $X \rightarrow \mathbf{0}$ is adm. epi

2) these two collections conform isomorphisms and are closed under composition.

3) • the collection of adm. monos is closed under cobase change:

$$\begin{array}{ll} \forall \text{ adm. mono } A \rightarrow B & A \rightarrow B \\ \forall \text{ morphism } A \rightarrow A' & \downarrow \text{to } \downarrow \\ \Rightarrow A' \amalg_B \underset{A}{\exists} \text{ exists and} & A' \rightarrow A' \amalg_B \\ A' \rightarrow A' \amalg_B \underset{A}{\exists} \text{ is an adm. mono.} & A. \end{array}$$

• the collection of adm. epis is closed under base change.

4) • if i is adm. mono, then $\text{coker}(i)$ is adm. epi and $i = \ker(\text{coker}(i))$

• if p is adm. epi, then $\ker(p)$ is adm. mono and $p = \text{coker}(\ker(p))$.

- Remarks
- 1) C exact $\Rightarrow C^{\text{op}}$ is exact.
 - 2) adm monos determine adm epis and conversely:
adm epis are precisely the cokernels of adm monos.

Def Let \mathcal{C} be an exact category. An exact sequence in \mathcal{C} is

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

where i adm mono, $p = \text{coker}(i)$
 p adm epi, $i = \text{ker}(p)$

$A \rightarrow B$
 $\downarrow \text{iso PB} \downarrow$
 $0 \rightarrow C$

Def A functor between exact categories is exact if:

- it preserves finite products (i.e. is additive)
- it preserves adm monos and adm epis
- it preserves pushouts along adm monos and pullbacks along adm epis.

(\Rightarrow it preserves exact sequences)

Examples

- 1) If A is an abelian category, then A is exact with
adm mono = mono, adm epi = epi.
- 2) If \mathcal{C} is an additive category, then \mathcal{C} has a minimal exact structure
with
adm mono = summand inclusions $A \rightarrow A \oplus B$
adm epi = summand projectives $A \oplus B \rightarrow A$.
e.g.: If R is a ring, $\text{Proj}(R)$.
- 3) X a scheme. The category $\text{Vect}(X)$ of finite locally free sheaves on X ,
has an exact structure with:
adm. epis = epimorphisms on stalks
adm. monos = kernels of adm epis.

A sequence in $\text{Vect}(X)$ is exact iff it is exact in $\text{Qcoh}(X)$.

Def \mathcal{C} exact cat. $K_0(\mathcal{C}) =$ free (abelian) group on iso. classes of objects of \mathcal{C}
modulo the relation $[B] = [A] + [C]$ for every
exact sequence $A \rightarrow B \rightarrow C$.

Goals: Define a space $K(\mathcal{C})$ for \mathcal{C} an exact category, such that:

- $\pi_0 K(\mathcal{C}) = K_0(\mathcal{C})$
- if \mathcal{C} has minimal exact structure, $K(\mathcal{C}) = |N(\mathcal{C}^{\approx}, \oplus)|^{op}$

The Q-construction for exact categories

Let \mathcal{C} be an exact cat. We define a category QC as follows:

objects: objects of \mathcal{C} .

morphisms from X to Y are spans

$$X \xleftarrow{p} Z \xrightarrow{i} Y$$

where p is adn epi and i is adn mono

(two spans $X \xleftarrow{z} Z \xrightarrow{t} Y$ are considered the same if there is an isomorphism $t \simeq t'$ over X and Y . (necessarily unique))

Identity: $X = X = X$.

Composition:

$$\begin{array}{ccc} X & \xleftarrow{p} & Y \\ \downarrow z & \nearrow t & \downarrow p \\ X & \xleftarrow{z'} & Y \\ & \xrightarrow{t'} & \\ & \xrightarrow{P} & \\ & \xrightarrow{T} & \\ & \xrightarrow{W} & \end{array} \rightsquigarrow X \xleftarrow{P} W.$$

lemma below

Easy: composition is associative.

Interpretation: "A morphism from X to Y in QC is an isomorphism of X with a subquotient of Y "

$$X \xleftarrow{p} Z \xrightarrow{i} Y \Leftrightarrow \text{two-step filtration on } Y:$$

$$\begin{array}{c} Y \xleftarrow{i} Z \xleftarrow{p} \ker(p) \\ \downarrow \quad \downarrow p \\ Y/\ker(p) \hookrightarrow X \\ \downarrow \\ Y/Z \end{array}$$

Rank $X \xleftarrow{p} Z \xrightarrow{i} Y$ is an iso in QC iff p and i are isomorphisms.

$$\underline{\text{Def}} \quad K(C) = \Omega_0 |N(QC)|$$

$$K_n(C) = \pi_n K(C) = \pi_{n+1} |N(QC)|$$

Remark This does not depend on the choice of $0 \in C$:

If $0 \xrightarrow{\sim} 0' \rightsquigarrow$ path in $|N(QC)|$
 \rightsquigarrow homotopy equivalence $\Omega_0 \simeq \Omega_0'$.

Functoriality: $f: C \rightarrow D$ exact functor

$$\rightsquigarrow QC \xrightarrow{Qf} QD \rightsquigarrow \Omega_0 |N(QC)| \rightarrow \Omega_{f(0)} |N(QD)|$$

$$\begin{matrix} \parallel \\ K(C) \end{matrix} \xrightarrow{K(f)} \begin{matrix} \parallel \\ K(D) \end{matrix}$$

natural isomorphism $\alpha: f \simeq g: C \rightarrow D$

$$\rightsquigarrow Q\alpha: Qf \simeq Qg$$

$$\rightsquigarrow K(C) \xrightarrow[K(f)]{K(\alpha)} K(D)$$

Properties

$$1) \quad K(C^\circ) \simeq K(C)$$

$$2) \quad K(C \times D) \simeq K(C) \times K(D), \quad K(\{0\}) \simeq *$$

3) if $(C_i)_{i \in I}$ is a filtered diagram of exact categories, then

$$\operatorname{colim}_i K_n(C_i) \xrightarrow{\sim} K_n(\operatorname{colim}_i C_i).$$

Cor C is an exact category, then $K(C)$ is an E_∞ -space.

Pf. $K: \text{ExCat} \rightarrow \text{Top}$ preserves finite products up to weak equivalence
 $\Rightarrow K$ preserves E_∞ -objects.

Every $C \in \text{ExCat}$ has a canonical E_∞ -structure:

$$\begin{array}{ccc} \text{Fin}_* & \longrightarrow & \text{Cat}, \quad I_+ \mapsto C(I) \simeq C^I. \\ & \downarrow \text{forget} & \text{(cf. exercises)} \\ & \triangleright \text{ExCat.} & \blacksquare \end{array}$$

Pf. 1) In fact, $Q(C^\circ) \simeq Q(C)$ (exercise)

2) $Q(C \times D) = Q(C) \times Q(D)$, $\Omega |N(-)|$ preserves finite products up to w.e.

$$3) Q(\text{colim}_i \mathcal{E}_i) \simeq \text{colim}_i Q(\mathcal{E}_i)$$

$$\left(\begin{array}{c} s\text{Set}_* \xrightarrow{\text{Ex}^\infty} \text{Kan}_* \\ \downarrow \pi_n = [\Delta^n / \partial\Delta^n, -]_* \\ \text{Set} \\ \cdot \text{ Ex}^\infty \text{ preserves filt. colims} \\ \cdot \left. \begin{array}{l} \Delta^n / \partial\Delta^n \\ \Delta^n / \partial\Delta^n \wedge \Delta^1_+ \end{array} \right\} \text{ compact in } s\text{Set} \end{array} \right) \xleftarrow{\quad} \left\{ \begin{array}{l} \pi_* \Omega [N(-)] \text{ preserves} \\ \text{ filtered colims, because} \\ \pi_* : s\text{Set}_* \rightarrow \text{Set} \\ \text{ preserves filtered colims} \end{array} \right\} \quad \square$$

Leftover lemma Consider a pullback square

$$\begin{array}{ccc} B & \xrightarrow{q} & C \\ j \downarrow & & \downarrow i \\ M & \xrightarrow{p} & N \end{array}$$

where i is adm. mono and p is adm. epi.

Then j is adm. mono and the square is also a pushout.

Proof

$$\begin{array}{ccc} B & \xrightarrow{q} & C \\ j \downarrow & \downarrow i & \\ M & \rightarrow & N \end{array} \quad Q = \text{coker}(i) \Rightarrow j \text{ is the kernel of } M \rightarrow Q$$

$$\begin{array}{ccc} & \downarrow & \downarrow \\ & Q & = Q \end{array} \quad \text{hence is adm. mono.}$$

$$\begin{array}{c} \text{ker}(p) = \text{ker}(j) \rightarrow 0 \\ \downarrow \quad \downarrow \\ B \xrightarrow{q} C \\ \downarrow \quad \downarrow \\ M \xrightarrow{p} N \end{array} \quad \Rightarrow \text{PO.} \quad \square$$