

Lemma Let \mathcal{C} be a category. There is an equivalence:

$$\left\{ \text{Covering spaces } / |N(\mathcal{C})| \right\} \simeq \underset{\downarrow}{\text{Fun}_{\text{inv}}(\mathcal{C}, \text{Set})} \subset \text{Fun}(\mathcal{C}, \text{Set})$$

Pr. Given a covering space $E \xrightarrow{p} |N(\mathcal{C})|$,
full subcategory
of functors sending every
morphism to an isomorphism.

Let $E_c = p^{-1}(c)$ for $c \in \mathcal{C}$. A morphism $\alpha: c \rightarrow c'$ in \mathcal{C} induces
 a path $c \xrightarrow{\alpha} c'$ in $|N(\mathcal{C})|$, hence $\alpha_*: E_c \xrightarrow{\sim} E_{c'}$.

So we get a functor $\mathcal{C} \rightarrow \text{Set}$, $c \mapsto E_c$.

Conversely, given $F: \mathcal{C} \rightarrow \text{Set}$, let $\int_{\mathcal{C}} F \xrightarrow{p_F} \mathcal{C}$ be the category of pairs
 (c, x) with $c \in \mathcal{C}$, $x \in F(c)$. Morphisms $(c, x) \rightarrow (c', x')$

are $c \xrightarrow{f} c'$ s.t. $F(f)(x) = x'$. The functor $\int_{\mathcal{C}} F \xrightarrow{p_F} \mathcal{C}$
 $(c, x) \mapsto c$

is a cocartesian fibration in sets: $p_F^{-1}(c) = F(c)$.

If $F: \mathcal{C} \rightarrow \text{Set}$ inverts all morphisms, then $|N(p_F)|$ is a covering map,
 by the previous proposition. □

Theorem Let \mathcal{C} be an exact category. Then $\begin{matrix} \pi_0 K(\mathcal{C}) & \simeq & K_0(\mathcal{C}) \\ \parallel & & \\ \pi_1 |N(\mathcal{C})| & & \end{matrix}$

Proof By the lemma, it suffices to construct an equivalence of categories

$$\begin{matrix} \text{Fun}_{\text{inv}}(Q\mathcal{C}, \text{Set}) & \simeq & K_0(\mathcal{C})-\text{Sets} \\ \overset{\cong}{F} & \longmapsto & F(\mathfrak{o}) \rtimes K_0(\mathcal{C}) \end{matrix}$$

$$\text{For } A \in \mathcal{C}: \quad i_A: \mathfrak{o} \xleftarrow{\cong} A \rightsquigarrow F(i_A): F(\mathfrak{o}) \rightarrow F(A)$$

$$p_A: \mathfrak{o} \xleftarrow{A} A \rightsquigarrow F(p_A): F(\mathfrak{o}) \rightarrow F(A)$$

$$\text{Let } A \text{ act on } F(\mathfrak{o}) \text{ by: } F(\mathfrak{o}) \xrightarrow{F(p_A)} F(A) \xrightarrow{F(i_A)^{-1}} F(\mathfrak{o})$$

→ this defines an action of the free group on $\pi_0(\mathcal{C}^{\leq 0})$ on $F(\mathfrak{o})$.

For an exact sequence $A \rightarrow B \rightarrow C$, we must check:

$$\begin{aligned} & (\text{def } c) (\text{def } o)^{-1} (\text{def } A) (\text{def } o)^{-1} \\ & \stackrel{?}{=} (\underbrace{\text{def } B}_{(\text{def } c)})(\underbrace{\text{def } o}_{(\text{def } B)})^{-1} \\ & (\text{def } c)(\text{def } B). \quad ((\text{def } o)(\text{def } A))^{-1} = (\text{def } A)^{-1} (\text{def } o)^{-1} \end{aligned}$$

$$\Leftrightarrow (\text{def } o)^{-1} (\text{def } A) \stackrel{?}{=} (\text{def } B)(\text{def } A)^{-1}$$

$$\Leftrightarrow (\text{def } A)(\text{def } A)^{-1} \stackrel{?}{=} (\text{def } o)^{-1} (\text{def } B)$$

$$\begin{array}{ccc} \text{def } A & \xrightarrow{\text{def } A} & \text{def } B \\ \downarrow & \text{def } A & \downarrow \\ \text{def } o & \xrightarrow{\text{def } A} & \text{def } B \\ \text{def } o & \xrightarrow{\text{def } A} & \text{def } B \end{array}$$

(1) key point

\Rightarrow We have defined an action of $K_0(C)$ on $F(o)$.

In the other direction: $\begin{array}{c} K_0(C) - \text{sets} \longrightarrow \text{functors}(QC, \text{Set}) \\ S \in K_0(C) \mapsto F_S \end{array}$

$$F_S(A) = S$$

$$F_S(A \xleftarrow{p} B \xrightarrow{i} C) = [\ker(p)]_{K_0(C)} : S \xrightarrow{\sim} S.$$

Easy to check:

- F_S is a functor

- $S \mapsto F_S$ is inverse to $F \mapsto F(o) \otimes K_0(C)$. \square

Proposition Let \mathcal{C} be an additive category with classes of "aden monos" and "aden epis". TFAE:

- 1) \mathcal{C} is exact
- 2) there exists a fully faithful $\overset{\text{additive}}{\hookrightarrow}$ embedding $\mathcal{C} \subset \mathcal{A}$ where \mathcal{A} is abelian, \mathcal{C} is closed under extensions in \mathcal{A} , and $\{\text{aden monos}\} = \{\text{mono in } \mathcal{A}\} \cap \mathcal{C}$ $\{\text{aden epis}\} = \{\text{epis in } \mathcal{A}\} \cap \mathcal{C}$.

Pf. See: Keller, "Chain complexes and stable categories"
Prop. A.2.

Quillen's Theorems A and B

Theorem A Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If $N(d \setminus f)$ is weakly contractible for all $d \in \mathcal{D}$, then $N(f): N(\mathcal{C}) \rightarrow N(\mathcal{D})$ is a weak equivalence.

Theorem B Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be a functor, such that, for every $d \rightarrow d'$ in \mathcal{D} , the induced morphism $N(d' \setminus f) \rightarrow N(d \setminus f)$ is a weak equivalence. Then, for every $d \in \mathcal{D}$, the canonical

$$\text{map } |N(d \setminus f)| \rightarrow \text{hofib}_d(|N(f)|)$$

is a weak equivalence

$$\begin{array}{ccc} & \uparrow & \\ d \setminus f & \xrightarrow{\text{forget}} & \mathcal{C} \\ \downarrow & \not\cong & \downarrow f \\ \ast & \xrightarrow{d} & \mathcal{D} \end{array}$$

Rmk Theorem B \Rightarrow Theorem A.

Examples of Thm A

- 1) If $p: \mathcal{E} \rightarrow \mathcal{B}$ is a Grothendieck pre(cw) fibration such that $N(p'(b))$ is weakly contractible for all $b \in \mathcal{B}$, then $N(p)$ is a w.e. Indeed $p'(b) \xleftarrow{\sim} b \setminus p \Rightarrow N(p'(b)) \simeq N(b \setminus p)$.

2) If $f: \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint $g: \mathcal{D} \rightarrow \mathcal{C}$, then

$d \setminus f = g(d) \setminus \mathcal{C}$ has an initial object $\Rightarrow N(d \setminus f)$ is w.e.

\Rightarrow Theorem A recovers the fact that $N(f)$ is a w.e.

Geometric realization of simplicial spaces

Ideas:

$$\begin{array}{ccc} \text{Fun}(\Delta^{\text{op}}, \text{Top})[\text{w.e.}] & \begin{matrix} \xleftarrow{\quad \text{const} \quad} \\ \xrightarrow{\quad \text{const} \quad} \end{matrix} & \text{Top}[\text{w.e.}] \\ \cup \\ \text{Fun}(\Delta^{\text{op}}, \text{Set}) & \xrightarrow{\quad \text{I.I} \quad} & \end{array}$$

$\text{I.I} = \text{hocolim}_{\Delta^{\text{op}}}$

Fact: The functor

$$(*) \quad \text{Fun}(\Delta^{\text{op}}, \text{sSet}) \simeq \text{Fun}(\Delta^n \times \Delta^n, \text{Set}) \xrightarrow{\text{diag}^*} \text{Fun}(\Delta^{\text{op}}, \text{Set}) = \text{sSet}$$

preserves weak equivalences, and the induced functor

$$\text{Fun}(\Delta^{\text{op}}, \text{sSet})[\text{w.e.}] \xrightarrow{\text{I.I}} \text{sSet}[\text{w.e.}]$$

is left adjoint to const.

Lemma Let $X_{\bullet\bullet}: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Top}/\text{sSet}$ be a bisimplicial space.

Then there are weak equivalences:

$$\left| p \mapsto |X_{p\bullet}| \right| \simeq \left| q \mapsto |X_{\bullet q}| \right| \simeq \left| n \mapsto X_{nn} \right|$$

Proof Obvious from (*):

$$\begin{array}{ccccc} \Delta^n \times \Delta^m & \xrightarrow{(\pi_1, \pi_2, \pi_3)} & \Delta^p \times \Delta^q \times \Delta^r & \longrightarrow & \text{Set} \\ \text{diag.} \uparrow & \text{diag.} \downarrow & \uparrow (\pi_1, \pi_2, \pi_3) & & \\ \Delta^{\text{op}} & \xrightarrow{\text{diag.}} & \Delta^m \times \Delta^q & \xrightarrow{\text{diag.}} & \Delta^p \times \Delta^r \end{array}$$

