

Proof of Theorem A $f: \mathcal{C} \rightarrow \mathcal{D}$.

Let $S(f)$ be the following category:

- objects are (X, Y, v) , $X \in \mathcal{C}$, $Y \in \mathcal{D}$, $v: Y \rightarrow f(X)$
- morphisms $(X, Y, v) \rightarrow (X', Y', v')$ is $X \xrightarrow{u} X'$ and $Y \xleftarrow{w} Y'$

such that

$$\begin{array}{ccc} Y & \xrightarrow{v} & f(X) \\ w \uparrow & \square & \downarrow f(u) \\ Y' & \xrightarrow{v'} & f(X') \end{array}$$

$$\rightsquigarrow S(f) \rightarrow \mathcal{C} \times \mathcal{D}^{\text{op}} \quad \begin{cases} \text{cofibered in sets, corresponding to} \\ (X, Y, v) \mapsto (X, Y) \\ \mathcal{C} \times \mathcal{D}^{\text{op}} \rightarrow \text{Set} \\ (X, Y) \mapsto \text{Hom}(Y, f(X)) \end{cases}$$

We have the diagram:

$$\begin{array}{ccccc} \mathcal{D}^{\text{op}} & \xleftarrow{P_2} & S(f) & \xrightarrow{P_1} & \mathcal{C} \\ \parallel & & \downarrow f' & & \downarrow f \\ \mathcal{D}^{\text{op}} & \xleftarrow{P_2} & S(\text{id}_{\mathcal{D}}) & \xrightarrow{P_1} & \mathcal{D} \end{array} \quad \text{where } f'(X, Y, v) = (f(X), Y, v)$$

\Rightarrow It suffices to show that $N(p_1)$ and $N(p_2)$ are weak equivalences

Define a bisimplicial set $T(f): \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Set}$:

$$T(f)_{pq} = \left\{ Y_p \rightarrow \cdots \rightarrow Y_0 \rightarrow f(X_0), X_0 \rightarrow \cdots \rightarrow X_q \right\}_{\text{in } \mathcal{D}}$$

$$T(f)_{\bullet q} = \coprod_{X_0 \rightarrow \cdots \rightarrow X_q} N(\mathcal{D}/f(X_0)^{\text{op}})$$

$$T(f)_{p\bullet} = \coprod_{Y_p \rightarrow \cdots \rightarrow Y_0} N(Y_0 \setminus f) \quad Y_0 \setminus f : (X \in \mathcal{C}, Y_0 \rightarrow f(X))$$

$$\text{diag}^* T(f) = N(S(f)).$$

$N(p_1)$ is w.e.: $T(f)_{pq} \xrightarrow{\varphi_{pq}} N(C)_q$ $\xrightarrow{\text{bisimplicial set constant in p-direction}}$

$$\text{diag}^*(\varphi): N(S(f)) \rightarrow N(C) \sqcup N(p_1)$$

Revisit w.r.t p :

$$\coprod_{X_0 \rightarrow \cdots \rightarrow X_q} [N(\mathcal{D}/f(X_0)^{\text{op}})] = |p \mapsto T(f)_{pq}| \xrightarrow{|\varphi_{pq}|} N(C)_q = \coprod_{X_0 \rightarrow \cdots \rightarrow X_q} p$$

$\mathcal{D}/f(x_0)$ has a final object $\Rightarrow N(\mathcal{D}/f(x_0)^{\text{op}})$ is weakly contractible

$\Rightarrow \{q_{\circ q}\}$ is a weak equivalence for all q

$\xrightarrow{\text{Lemma}}$ $\text{diag}^*(q)$ is a weak equivalence
 $\overset{''}{N}(p_1)$

$N(p_2)$ is a w.e.: $T(f)_{pq} \xrightarrow{\Psi_{pq}} N(\mathcal{D}^{\text{op}})_p$

$$\text{diag}^*(\eta_f) = N(p_2) : N(s(f)) \rightarrow N(\mathcal{D}^{\text{op}})$$

Realize η_f in the q -direction:

$$\coprod_{Y_p \rightarrow Y_0} |N(Y_0 \setminus f)| = |\{q \mapsto T(f)_{pq}\} \xrightarrow{|\Psi_{pq}|} N(\mathcal{D}^{\text{op}})_p = \coprod_{Y_p \rightarrow Y_0} p_t$$

By assumption, $N(Y_0 \setminus f)$ is weakly contractible

$\Rightarrow \{q_{\circ p_0}\}$ is a weak equivalence
 $\xrightarrow{\text{Lemma}}$ $\text{diag}^*(\eta_f)$ is a weak equivalence
 $\overset{''}{N}(p_2)$.



Pf of Theorem B: See Quillen, Higher alg. K-theory I.

The fundamental theorems of the Q-construction

| | | |
|----------------------------------|---|---------|
| Additivity theorem | } | HAKT I |
| Resolution theorem | | |
| Dévissage theorem | | |
| Localization theorem | | |
| Cofinality theorem | } | HAKT II |
| Comparison with group completion | | |

Let \mathcal{C} be an exact category, let $\mathcal{EC} \subset \text{Fun}([2], \mathcal{C})$ be the full subcategory of exact sequences. There are 3 functors

$$\mathcal{EC} \xrightarrow{s} \mathcal{C} \quad \text{with natural transformations } s \rightarrow t \rightarrow q .$$

\xrightarrow{t} \xrightarrow{q}

such that $\forall E \in \mathcal{EC}$, $s(E) \rightarrow t(E) \rightarrow q(E)$
 is an exact sequence.

\mathcal{EC} has an exact structure if f is adic mono/epi iff $s(f), t(f), g(f)$ are in \mathcal{C} .

e.g.: $A \rightarrow B \rightarrow C$
 $\downarrow \quad \downarrow \quad \downarrow$
 $A' \rightarrow B' \rightarrow C'$

$\Rightarrow s, t, g$ are exact functors.

Theorem (Additivity theorem)

\mathcal{C} exact category. Then $K(\mathcal{EC}) \xrightarrow{(s, g)} K(\mathcal{C}) \times K(\mathcal{C})$

is a weak equivalence.

Pf. It suffices to show that $N(Q(\mathcal{EC})) \xrightarrow{(s, g)} N(Q\mathcal{C}) \times N(Q\mathcal{C})$ is a weak equivalence. By Theorem A, it suffices to show that $N((s, g)/(X, Y))$ is weakly contractible for all $X, Y \in \mathcal{C}$. where $(s, g): Q(\mathcal{EC}) \rightarrow Q(\mathcal{C}) \times Q(\mathcal{C})$

$A = (s, g)/(X, Y)$:

objects are triples (E, u, v) , $E \in Q(\mathcal{EC})$

$u: s(E) \rightarrow X$ and $v: g(E) \rightarrow Y$
morphisms in $Q(\mathcal{C})$

$A' \subset A$: full subcategory where u is $\begin{smallmatrix} \leftarrow \\ \searrow \end{smallmatrix}$

$A'' \subset A'$: _____ v is $\begin{smallmatrix} \leftarrow \\ \searrow \end{smallmatrix}$

The contractibility of $N(A)$ follows from the following claims:

Claim 1 $A'' \subset A' \subset A$ have adjoints

Claim 2 A'' has an initial objects.

Pf of claim 2: the zero sequence $0 \rightarrow 0 \rightarrow 0$

with $0 \xleftarrow{X} X$ and $0 \xrightarrow{Y} Y$

is an initial object in A'' .

A general object in A'' is: $A \rightarrow B \rightarrow C$ $A \xleftarrow{X} X$ $C \xrightarrow{Y} Y$

A morphism from the 1st to the 2nd is:

- a morphism from O to A over X :

$$\Rightarrow A' \xrightarrow{\sim} A.$$

= there is a unique
such morphism.

$$\begin{array}{ccccc} O & \leftarrow & X = X & & \\ \uparrow & & \nearrow \text{PB} & \parallel & \parallel \\ A' & \xrightarrow{\sim} & A & & \\ \downarrow & & \searrow \text{PB} & & \\ A & \leftarrow & X & & \end{array}$$

- similarly, $\exists!$ morphism from O to C over Y
- hence the claim.

Pf of claim 1: $A' \subset A$ has a left adjoint

$$A \rightarrowtail B \twoheadrightarrow C, \quad A \leftarrow^X \underset{\text{PO}}{\downarrow} X, \quad C \leftarrow^{Y'} \underset{\text{PO}}{\downarrow} Y.$$

$$\begin{array}{ccc} A \rightarrowtail B \twoheadrightarrow C & & \\ \downarrow & \text{PO} & \downarrow \cong \\ A' \rightarrowtail B' \twoheadrightarrow C' & & \\ & & A' \leftarrow^X = X \quad C \leftarrow^{Y'} \underset{\text{PO}}{\downarrow} Y. \end{array}$$

universal morphism to an object in A' .

- $A'' \subset A'$ has a right adjoint, similarly. \blacksquare