

K_0

$\text{CMon} = \underline{\text{category of commutative monoids}}$
 $\text{Ab} = \underline{\text{abelian groups}}$

The inclusion $\text{Ab} \hookrightarrow \text{CMon}$ has a left adjoint $M \mapsto M^{\text{gp}}$, called group completion. (or Grathendieck group).

Constructions: 1. $M^{\text{gp}} = \mathbb{Z}[M] / \text{subgp generated by } \{m+n\} - \{m\} - \{n\}$
 2. $M^{\text{gp}} = M \times M / \begin{cases} (m,n) \sim (m',n') \\ \text{if } \exists t \in M \text{ s.t. } m+n+t = m'+n+t. \end{cases}$

Notation: R ring, $\text{Proj}(R) = \text{category of f.g. projective (left) } R\text{-modules}$.

C category $\rightsquigarrow C^\sim$ maximal subgroupoid of C .

C groupoid $\rightsquigarrow \pi_0 C = \text{set of isomorphism classes of objects of } C$.

Def R ring. $K_0(R) = \underbrace{\pi_0(\text{Proj}(R)^\sim)^{\text{gp}}}_{\hookrightarrow \text{comm. monoid under } \oplus}$

Rank Recall an R -module P is f.g. proj. iff it is a summand of R^n for some $n \geq 0$.
 Hence, every element in $K_0(R)$ is of the form $P - Q$ for some $P \in \text{Proj}(R)$ and $n \geq 0$.

$P - R^n$ for some $P \in \text{Proj}(R)$ and $n \geq 0$.

Indeed, $P - \underbrace{Q}_{Q \oplus Q' \cong R^n} \Rightarrow P - Q = P \oplus Q' - R^n \text{ in } K_0(R)$.

Example

- $(\mathbb{N}, +)^{\text{gp}} = (\mathbb{Z}, +)$
 - $(\mathbb{N} \setminus \{0\}, \cdot)^{\text{gp}} = (\mathbb{Q}_{>0}, \cdot)$
 - $\{ \text{iso. class of countably generated proj. } R\text{-modules} \}^{\text{gp}} = 0$
- $$R^\infty = \bigoplus_{n \in \mathbb{N}} R, \quad M \quad M \oplus N \cong R^\infty$$

"the Eilenberg swindle"

$$\begin{aligned} M \oplus R^\infty &= M \oplus R^\infty \oplus R^\infty \oplus \dots \\ &\cong M \oplus (N \oplus M) \oplus (N \oplus M) \oplus \dots \\ &\cong (M \oplus N) \oplus (M \oplus N) \oplus \dots \\ &\cong R^\infty \oplus R^\infty \oplus \dots \\ &\cong R^\infty \end{aligned}$$

$\Rightarrow M = 0$ in the group completion.

- If R is comm. and every f.g.-proj R -module is free, then $K_0(R) = \mathbb{Z}$:
 - R is a PID
 - $k(x_1, \dots, x_n)$ where k is a PID (Quillen-Suslin theorem)
 - R is a local ring (Nakayama's lemma).
- D division ring, $K_0(\text{Mat}_n(D)) \cong \mathbb{Z}$
 Every $\text{Mat}_n(D)$ -module is a sum of copies of simple module
 D^n (Morita theory).

Exercise: if $(M, +, \cdot)$ is a semiring, then $M^{\otimes P}$ has a structure of ring

$$\Rightarrow \text{if } R \text{ is commutative, } \pi_0(P_{\text{fg}}(R^{\otimes P})) \text{ is a semiring under } \oplus, \otimes$$

$$\Rightarrow K_0(R) \text{ is a comm. ring.}$$

Example G finite group.

- The Burnside ring of G is $A(G) = \pi_0(\text{finite } G\text{-sets})^{\otimes P}$
 As a group: $A(G) = \bigoplus_{\substack{\text{conj. classes} \\ \text{of subgroups of } G}} \mathbb{Z}$
- The representation ring of G is $R(G) = \pi_0(\text{fin.dim. } G\text{-representations})^{\otimes P}$
 As a group: $R(G) = \bigoplus_{\substack{\text{conj. classes} \\ \text{of elements of } G}} \mathbb{Z}$ (Maschke's theorem)
 \oplus, \otimes
 $K_0(\mathbb{C}[G]) \cong R(G)$
 $\cong \text{ab. group.}$

Q: How do we understand $M^{\otimes P}$?

Ans: Find interesting morphisms $M \rightarrow A$ where A is an abelian group
 $\downarrow M^{\otimes P}$

In the context of K-theory, $K_0(R) \rightarrow A$ is called a trace map.

Trace maps

The rank map (R commutative)

Recall: a f.g.-proj R -module P has a rank $\text{rk}(P): \text{Spec}(R) \rightarrow \mathbb{N}$
 $x \mapsto \dim_{K(x)} P \otimes_R K(x)$

$$\kappa(p) = \text{Frac}(R/p) \quad (P \otimes_R \kappa(x))$$

If $\text{Spec}(R)$ is connected ($\Leftrightarrow R$ has exactly two idempotents 0,1)
then $\text{rk}(P)$ is constant.

$$\begin{aligned}\text{rk}(P \oplus Q) &= \text{rk}(P) + \text{rk}(Q). \\ \text{rk}_R(P \otimes Q) &= \text{rk}(P) \text{rk}(Q)\end{aligned}$$

$$\begin{array}{ccc} \pi_0(\text{Proj}(R)^\simeq) & \xrightarrow{\text{rk}} & \text{Maps}(\text{Spec } R, \mathbb{N}) \\ \downarrow & & \downarrow \text{continuous maps} \\ K_0(R) & \xrightarrow[\exists! \text{ rk}]{} & \text{Maps}(\text{Spec } R, \mathbb{Z}) \quad \text{morphism of comm rings.} \end{array}$$

Exercise: if X is a topological space, $\text{Maps}(X, \mathbb{N})^{\oplus P} \simeq \text{Maps}(X, \mathbb{Z})$.

Given $n: \text{Spec } R \rightarrow \mathbb{N}$, we can define a fg-proj R -module R^n

$$\begin{aligned} &\rightsquigarrow R \simeq \prod_{m \in \mathbb{N}} R_m \quad \rightsquigarrow R^n = \prod_{m \in \mathbb{N}} R_m^n. \\ &\text{finite product} \\ &\rightsquigarrow \text{Maps}(\text{Spec } R, \mathbb{N}) \xrightarrow{n \mapsto R^n} \pi_0(\text{Proj}(R)^\simeq) \xrightarrow{\text{rk}} \text{Maps}(\text{Spec } R, \mathbb{N}) \\ &\xrightarrow{(-)^{\oplus P}} \text{Maps}(\text{Spec } R, \mathbb{Z}) \xrightarrow{\text{id}} K_0(R) \xrightarrow[\text{id}]{} \text{Maps}(\text{Spec } R, \mathbb{Z}) \\ &\Rightarrow K_0(R) \simeq \text{Maps}(\text{Spec } R, \mathbb{Z}) \oplus \tilde{K}_0(R) \\ &\text{where } \tilde{K}_0(R) = \ker(\text{rk}) \text{ an ideal in } K_0(R) \end{aligned}$$

For $n \geq 0$, $\text{Proj}_n(R) \subset \text{Proj}(R)$ subcategory of fgproj R -modules of rank n .

$$\begin{array}{ccc} \pi_0(\text{Proj}_n(R)^\simeq) & \longrightarrow & \tilde{K}_0(R) \subset K_0(R) \\ P & \longmapsto & P - R^n \end{array}$$

$$\begin{array}{ccc} \text{Proj}_n(R) & \longrightarrow & \text{Proj}_{n+1}(R). \\ P & \longmapsto & P \oplus R \end{array}$$

$$\Rightarrow \text{get a map } \varinjlim_{n \rightarrow \infty} \pi_0(\text{Proj}_n(R)^\simeq) \longrightarrow \tilde{K}_0(R).$$

Prop. This map is a bijection.

Pf. Surjectivity: use that every element in $K_0(R)$ is $P - R^n$. ✓

Injectivity: $P - R^n = Q - R^m$ in $\tilde{K}_0(R)$.

$$\exists T \in \text{Proj}(R) \text{ s.t. } P \oplus R^n \oplus T \simeq Q \oplus R^m \oplus T.$$

$$\exists T' \text{ s.t. } T \oplus T' \simeq R^k \Rightarrow P \oplus R^{n+k} \simeq Q \oplus R^{m+k}$$