

Def A noetherian scheme X is regular if for every $x \in X$, $\mathcal{O}_{X,x}$ is regular ($\Leftrightarrow \mathcal{O}(U)$ is regular for every affine open $U \subset X$).

Examples.

- if X is smooth over a field k , then X is regular. The converse holds if k is perfect and X is of finite type / k .
- k field, $R = k[x]/(x^2)$. Then the R -module k has no finite projective resolution $\Rightarrow R$ is not regular. (also: $\dim(R) = 0$, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$)

Recall: if $\mathcal{F} \rightarrow \mathcal{G}$ is an epimorphism between finite locally free sheaves on a scheme X , then the kernel is finite locally free.

So $\text{Vect}(X) \subset \text{Coh}(X)$ satisfies assumption a) of the theorem.

What about b)?

Def. A scheme X has the resolution property if for every f.g. quasi-coherent sheaf \mathcal{F} , there exists a vector bundle \mathcal{V} and a surjection $\mathcal{V} \rightarrow \mathcal{F}$.

Example (Serre) A quasi-projective scheme over a ring has the resolution property ($\exists \mathcal{O}(i) \rightarrow \mathcal{F}$ for $i \gg 0$).

Theorem (Illusie) Let X be a separated regular noetherian scheme. Then X has the resolution property. Moreover, $\text{Vect}(X)_{\infty} = \text{Coh}(X)$.

In particular, assumption b) holds for separated regular noetherian schemes.

Corollary: If X is a separated regular noetherian scheme, then

$$K(\text{Vect}(X)) \xrightarrow{\sim} K(\text{Coh}(X)).$$

Remark. The affine plane with doubled origin X over a field k is an example of a regular noeth. scheme that does not have the resolution property. Moreover:

$$K(\text{Vect}(X)) \simeq K(k)$$

$$K(\text{Coh}(X)) \simeq K(k) \times K(k).$$

(see Thomason-Trobaugh, exercise 8.6)

Notation: If X is a scheme, $K^{\text{naive}}(X) := K(\text{Vect}(X))$.

If X is noetherian: $G(X) := K(\text{Coh}(X))$.
 \parallel
 $K'(X)$

Remark $K^{\text{naive}}(X)$ is a well-behaved invariant of X if X has the resolution property, but not otherwise.

The official definition of $K(X)$ uses Waldhausen S_2 -construction, and it is such that $K(X) \cong G(X)$ for any regular noetherian scheme X .

Let's go back to the general situation $\mathcal{P} \subset \mathcal{E}$.

Lemma Suppose a) & b) from the resolution theorem hold. Then:

1) $\mathcal{P}_n \subset \mathcal{E}$ is closed under extensions

2) if $X \rightarrow Y \rightarrow Z$ is exact in \mathcal{E} with $Y \in \mathcal{P}_n$ and $Z \in \mathcal{P}_{n+1}$, then $X \in \mathcal{P}_n$.

Pf. exercise.

Proof of the resolution theorem

$$\mathcal{P}_\infty = \bigcup_n \mathcal{P}_n \implies K_*(\mathcal{P}_\infty) = \text{colim}_n K_*(\mathcal{P}_n)$$

So it suffices to show that $K(\mathcal{P}_n) \rightarrow K(\mathcal{P}_{n+1})$ is a weak equivalence for all $n \geq 0$.
 $(\mathcal{P}_0 = \mathcal{P})$

We have: (1) if $X \rightarrow Y$ is an adm. mono in \mathcal{P}_{n+1} with $Y \in \mathcal{P}_n$, then $X \in \mathcal{P}_n$

(2) For any $Z \in \mathcal{P}_{n+1}$, there exists an adm. epi

$$P \rightarrow Z \text{ in } \mathcal{P}_{n+1} \text{ with } P \in \mathcal{P}_n.$$

[(1) by Lemma, and (2) follows from Lemma and b)]

Consider the factorization:

$$\begin{array}{ccc} \mathcal{Q}\mathcal{P}_n & \longrightarrow & \mathcal{Q}\mathcal{P}_{n+1} \\ \text{ess.} \searrow \downarrow \text{surjective } g & & \uparrow \text{fully faithful } f \\ & \mathcal{A} & \end{array}$$

We prove that $N(f)$ and $N(g)$ are weak equivalences using Theorem A.

$N(g)$ is w.c. We show $N(g/P) \simeq *$ for all $P \in \mathcal{A}$ (i.e. $P \in \mathcal{P}_n$)

g/P : objects are $Q \in \mathcal{P}_n$ with $Q \leftarrow R_0 \rightarrow R_1$ in $\mathcal{Q}\mathcal{P}_{n+1}$

$\Rightarrow g/P$ is a poset of pairs (R_0, R_1) of \mathcal{P}_{n+1} -admissible subobjects $R_i \subset R_0 \subset P$ s.t. $R_0/R_1 \in \mathcal{P}_n$

with $(R_0, R_1) \leq (R'_0, R'_1)$ iff

$$R'_0 \supset R_0 \supset R_1 \supset R'_1.$$

By (1), $R_0 \in \mathcal{P}_n$, w $(R_0, 0) \in g/P$. We have:

$$\begin{array}{ccccc} (R_0, R_1) & \leq & (R_0, 0) & \geq & (0, 0) \\ & \swarrow \text{id} & \uparrow & \searrow \text{const} & \\ & & (R_0, R_1) & & \end{array}$$

\Rightarrow zig-zag of natural transformation between $\text{id}_{g/P}$ and a constant functor.

$\Rightarrow N(g/P)$ is weakly contractible.

$N(f)$ is a w.c. $f: \mathcal{A} \hookrightarrow \mathcal{Q}\mathcal{P}_{n+1}$ full subcategory on \mathcal{P}_n .

We show that $N(X|f)$ is weakly contractible for every $X \in \mathcal{P}_{n+1}$.

$\mathcal{F} := X|f$: objects are pairs (P, u) where $P \in \mathcal{P}_n$ and

$$u: X \leftarrow P' \rightarrow P \text{ in } \mathcal{Q}\mathcal{P}_{n+1} \text{ (so } P' \in \mathcal{P}_n \text{ by (1))}$$

Let $\mathcal{F}' \subset \mathcal{F}$ be the full subcategory where $P' = P$.

The inclusion $\mathcal{F}' \subset \mathcal{F}$ has a right adjoint sending (P, u)

to $(P', X \leftarrow P' \rightarrow P)$.

It remains to show that $N(\mathcal{F}') \simeq *$.

\mathcal{F}' : • objects are pairs (P, u) where $P \in \mathcal{P}_n$ and $u: P \twoheadrightarrow X$ is a \mathcal{P}_{n+1} -admissible epi.

• a morphism $(P, u) \rightarrow (P', u')$ is a \mathcal{P}_{n+1} -admissible epi

$$\begin{array}{ccc} P & \longleftarrow & P' \\ & \searrow u & \swarrow u' \\ & X & \end{array}$$

Assertion (2) $\Leftrightarrow \mathcal{F}'$ is nonempty.

Also: if $(P, u), (P', u') \in \mathcal{F}'$, then $(P \times_X P', P \times_X P' \twoheadrightarrow X)$ is also in \mathcal{F}' : we have an exact sequence

$$\ker(u) \twoheadrightarrow P \times_X P' \twoheadrightarrow P' \quad \text{and } \ker(u) \in \mathcal{P}_n \text{ by (4)}$$

\mathcal{P}_n closed under extensions $\Rightarrow P \times_X P' \in \mathcal{P}_n$.

Let's fix $(P', u') \in \mathcal{F}'$. Then:

$$\begin{array}{ccccc} (P \twoheadrightarrow X) & \longrightarrow & (P \times_X P' \twoheadrightarrow X) & \longleftarrow & (P' \xrightarrow{u'} X) \\ & \searrow \text{id} & \uparrow & \swarrow \text{const} & \\ & & (P \twoheadrightarrow X) & & \end{array}$$

\leadsto we have a zig-zag of natural transformations between $\text{id}_{\mathcal{F}'}$ and a constant functor.

$\Rightarrow N(\mathcal{F}')$ is weakly contractible. \square