

### Dévissage

A abelian category,  $\mathcal{B} \subset \mathcal{A}$  full subcategory closed under finite products, subobjects, and quotients. Then  $\mathcal{B}$  is abelian and the inclusion  $\mathcal{B} \hookrightarrow \mathcal{A}$  is exact, and  $Q\mathcal{B} \subset Q\mathcal{A}$  full subcategory spanned by  $\mathcal{B}$ .

### Theorem (Dévissage)

Let  $\mathcal{B} \subset \mathcal{A}$  as above. Suppose that every  $A \in \mathcal{A}$  admits a finite filtration  $0 = A_0 \subset A_1 \subset \dots \subset A_n = A$  such that  $A_i/A_{i-1} \in \mathcal{B}$ . Then  $K(\mathcal{B}) \xrightarrow{\sim} K(\mathcal{A})$ .

Remark  $\mathcal{B}$  is not necessarily closed under extensions in  $\mathcal{A}$ ! (if it is, then  $\mathcal{B} = \mathcal{A}$ ).

### Main example (ex. 10).

$X$  a noetherian scheme,  $Z \xrightarrow{i} X$  closed immersion

Let  $\text{Coh}_Z(X) \subset \text{Coh}(X)$  full subcategory of sheaves supported on  $Z$  (i.e.  $F|_{X-Z} = 0$ ). Then

$$i_* : \text{Coh}(Z) \hookrightarrow \text{Coh}_Z(X)$$

is fully faithful and satisfies the assumptions of the dévissage theorem.

$$\text{Hence } K(\text{Coh}(Z)) \xrightarrow{\sim} K(\text{Coh}_Z(X))$$

$$\begin{array}{ccc} \parallel & & \parallel \\ G(Z) & & G_Z(X) = G(X \text{ on } Z) \end{array}$$

e.g.:  $X = \text{Spec } \mathbb{Z}$ ,  $Z = \text{Spec } \mathbb{F}_p$ .  $M \in \text{Coh}(X)$  is supported on  $Z$   
iff  $M[\mathbb{F}_p] = 0$

$$\rightarrow K(\text{Proj}(\mathbb{F}_p)) \simeq K(\text{$p$-power torsion f.g. abelian groups}).$$

Proof of dévissage let  $f : Q\mathcal{B} \hookrightarrow Q\mathcal{A}$ .

By Theorem A, it suffices to show  $N(f/A)$  is weakly contractible for every  $A \in Q\mathcal{A}$ .

As in the proof of resolution,  $f/A$  is equivalent to the poset  $\gamma(A)$  of pairs  $(A_0, A_1)$  of subobjects  $A_0 \subset A_1 \subset A$  s.t.  $A_1/A_0 \in \mathcal{B}$ , with  $(A_0, A_1) \leq (A'_0, A'_1)$  iff  $A'_0 \subset A_0 \subset A_1 \subset A'_1$ .

By assumption, there is  $0 = A_0 \subset A_1 \subset \dots \subset A_n = A$  with  $A_i/A_{i-1} \in \mathcal{B}$ .

By induction on  $n$ , it suffices to show:

Claim: If  $A' \subset A$  s.t.  $A/A' \in \mathcal{B}$ , then  $N(\gamma(A')) \rightarrow N(\gamma(A))$  is a weak equivalence.

Define functors:

$$r: \gamma(A) \rightarrow \gamma(A'), \quad (A_0, A_1) \mapsto (A_0 \cap A', A_1 \cap A')$$

$$s: \gamma(A) \rightarrow \gamma(A), \quad (A_0, A_1) \mapsto (A_0 \cap A', A_1)$$

Well-defined: •  $(A_1 \cap A') / (A_0 \cap A')$  is a subobject of  $A_1/A_0 \in \mathcal{B}$   
•  $A_1 / (A_0 \cap A')$  is a subobject of  $A_1/A_0 \oplus A/A' \in \mathcal{B}$

$$\left[ \begin{array}{l} \ker(A_1 / (A_0 \cap A') \rightarrow A_1/A_0) = A_0 / (A_0 \cap A') \\ \ker(A_1 / (A_0 \cap A') \rightarrow A/A') = (A_1 \cap A') / (A_0 \cap A') \end{array} \right]$$

Let  $i: \gamma(A') \rightarrow \gamma(A)$ . Note that  $r \circ i = \text{id}_{\gamma(A')}$

We have natural transformations

$$i \circ r \rightarrow s \leftarrow \text{id}_{\gamma(A)}$$

$$(A_0 \cap A', A_1 \cap A') \in (A_0 \cap A', A_1) \ni (A_0, A_1)$$

$\Rightarrow N(i \circ r)$  is homotopic to  $\text{id}_{N(\gamma(A))}$

$\Rightarrow N(i)$  is a weak equivalence □

Corollary Let  $A$  be an abelian category where every object has finite length (i.e.  $\exists 0 = A_0 \subset A_1 \subset \dots \subset A_n = A$  with  $A_i/A_{i-1}$  is simple)

$$\text{Then } K_*(A) \cong \bigoplus_S K_*(\text{End}(S))$$

$\uparrow$  division rings

where  $S$  ranges over a set of representatives of iso classes of simple objects.

Proof. Let  $\mathcal{B} \subset \mathcal{A}$  be the subcategory of semi-simple objects  
(i.e. finite sums of simple objects).

$$\Rightarrow K(\mathcal{B}) \xrightarrow{\sim} K(\mathcal{A}) \text{ by devissage.}$$

For  $I$  a finite set of simple objects, let  $\mathcal{B}_I \subset \mathcal{B}$  be the subcategory generated by  $I$  under finite sums. Then  $\mathcal{B} = \text{colim}_I \mathcal{B}_I$  (filtered) and  $\mathcal{B}_I \xleftarrow[\oplus]{\cong} \prod_{S \in I} \mathcal{B}_{\{S\}}$  (fully faithful because  $\text{Hom}(S_i, S_j) = 0$  if  $S_i \neq S_j$ )

$$\Rightarrow K(\mathcal{B}_I) \simeq \prod_{S \in I} K(\mathcal{B}_{\{S\}})$$

Finally, there is an equivalence  $\mathcal{B}_{\{S\}} \simeq \text{Proj}(\text{End}(S)^{\text{op}})$   
(right  $\text{End}(S)$ -modules)  
 $M \mapsto \text{Hom}(S, M)$

$$\Rightarrow K(\mathcal{B}_{\{S\}}) = K(\text{End}(S))$$



Corollary. Let  $X$  be a noetherian scheme,  $Z \subset X$  a closed subscheme defined by a nilpotent ideal. Then  $G(Z) \simeq G(X)$ .

Proof. Apply devissage to  $\text{Coh}(Z) \hookrightarrow \text{Col}_Z(X) \underset{\substack{\uparrow \\ X-Z=\emptyset}}{=} \text{Col}(X)$ . □

"G-theory is nilinvariant".

Warning: This fails for K-theory. Although  $K_0$  is nilinvariant,  $K_i$  is not for  $i \geq 1$ . e.g.:  $K_1(k) \not\simeq K_1(k[x]/(x^2))$ ,  
However, the difference between  $K(Z)$  and  $K(X)$  is well understood and "simple" (measured by topological cyclic homology)

### Localization

Let  $\mathcal{A}$  be an abelian category.

A Serre subcategory  $\mathcal{B} \subset \mathcal{A}$  is a full subcategory closed under finite products, subobjects, quotients, and extensions.

Then there exists an abelian category  $\mathcal{A}/\mathcal{B}$  and an exact functor  $\mathcal{A} \xrightarrow{s} \mathcal{A}/\mathcal{B}$  with the following universal property:

$\forall$  abelian category  $\mathcal{C}$ ,

$$\text{Fun}_{\text{ex}}(\mathcal{A}/\mathcal{B}, \mathcal{C}) \xrightarrow{s^*} \text{Fun}_{\text{ex}}(\mathcal{A}, \mathcal{C})$$

is fully faithful and  $F: \mathcal{A} \rightarrow \mathcal{C}$  is in the essential image  
iff  $F|_{\mathcal{B}} \simeq 0$ .

(Gabriel, "Des catégories abéliennes")

Moreover,  $\mathcal{B} = \{A \in \mathcal{A} \mid s(A) \simeq 0\}$ .

Explicit construction of  $\mathcal{A}/\mathcal{B}$ : objects : same as in  $\mathcal{A}$ .

morphisms:  $\text{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \underset{\substack{X' \subset X \\ Y' \subset Y \\ \text{r.t. } X/X' \in \mathcal{B} \\ \text{and } Y'/Y \in \mathcal{B}}}{\text{colim}} \text{Hom}_{\mathcal{A}}(X', Y/Y')$

So any morphism  $X \rightarrow Y$  in  $\mathcal{A}/\mathcal{B}$  factors as

$$X \xrightarrow{\quad} X' \rightarrow Y/Y' \leftarrow Y.$$

Example.  $X$  a noetherian scheme,  $Z \subset X$  closed subscheme,  
 $U \subset X$  open complement.

Then  $\text{Coh}_Z(X) \subset \text{Coh}(X)$  is a Serre subcategory

and  $\text{Coh}(X)/\text{Coh}_Z(X) \xrightarrow{\sim} \text{Coh}(U)$ ,  
 $F \mapsto F|_U$ .

(see exercises).