

Theorem (Localization)

Let $B \subset A$ be a Serre subcategory. Then there is a homotopy fiber sequence

$$K(B) \rightarrow K(A) \rightarrow K(A/B)$$

$$\text{i.e.: } K(B) \xrightarrow{\sim} \text{hofib}_0(K(A) \rightarrow K(A/B)).$$

Hence, there is a long exact sequence

$$\cdots \rightarrow K_{i+1}(A/B) \xrightarrow{\partial} K_i(B) \rightarrow K_i(A) \rightarrow K_i(A/B) \rightarrow \cdots$$

Corollary X noeth. scheme, $Z \hookrightarrow X$ closed immersion, $j: U \hookrightarrow X$ open complement.

1) There is homotopy fiber sequence $G(Z) \xrightarrow{i^*} G(X) \xrightarrow{j^*} G(U)$

2) If X and Z are regular and separated, then there is a homotopy fiber sequence $K^{\text{naive}}(Z) \rightarrow K^{\text{naive}}(X) \rightarrow K^{\text{naive}}(U)$.

Pf. 1) follows from localization + devissage.

2) follows from 1) and resolution theorem □

Idea of the proof of Localization

Since Ω^2 preserves homotopy fiber sequences (cf. exercise), it suffices to show that

$$|N(\Omega B)| \simeq \text{hofib}\left(|N(\Omega A)| \xrightarrow{\Omega s} |N(\Omega(A/B))|\right)$$

For this, we use Theorem B: It suffices to show:

1) for every $u: V' \rightarrow V$ in $\Omega(A/B)$,

$$u^*: V/Q_S \rightarrow V'/Q_S \text{ is a weak equivalence}$$

2) $\Omega B \hookrightarrow \Omega Q_S$ is a weak equivalence (use Theorem A).

Cofinality theorem

Let \mathcal{C} be an exact category, $\mathcal{C}_0 \subset \mathcal{C}$ a full subcategory closed under extensions. Suppose: $\forall X \in \mathcal{C}, \exists Y \in \mathcal{C}$ such that $X \oplus Y \in \mathcal{C}_0$.

Then $K_0(\mathcal{C}_0) \rightarrow K_0(\mathcal{C})$ is injective (cf. exercise)

and $K_i(\mathcal{C}_0) \xrightarrow{\sim} K_i(\mathcal{C})$ for $i \geq 1$.

Remark This was proved by Gertsen under additional assumptions (which hold if \mathcal{C} has minimal exact structure). The general statement is due to Waldhausen, who proved a more general cofinality theorem for the S_- -construction.

Example: R ring. $\text{Free}(R) \subset \text{Proj}(R)$. Then:

- $K_0(\text{Free}(R)) \hookrightarrow K_0(\text{Proj}(R))$ cyclic subgroup generated by $[R]$.
- $K_i(\text{Free}(R)) \xrightarrow{\sim} K_i(\text{Proj}(R))$ for $i \geq 1$.

Comparison with group completion

Let \mathcal{C} be an additive category, with minimal exact structure. Then there is weak equivalence of E_∞ -spaces

$$|N(\mathcal{C}^{\approx, \oplus})|^{gp} \xrightarrow{\sim} K(\mathcal{C}).$$

(see: Grayson, HAKT II)

Tor-dimension and functoriality

$f: R \rightarrow S$ morphism of comm rings

$$\begin{aligned} f^*: \text{Proj}(R) &\rightarrow \text{Proj}(S) & \rightsquigarrow f^*: K(R) &\rightarrow K(S) \\ M &\mapsto M \otimes_R S \end{aligned}$$

What about G -theory? (R, S noetherian)

$$\begin{aligned} f^*: \text{Col}(R) &\rightarrow \text{Col}(S) & \text{is not exact in general.} \\ M &\mapsto M \otimes_R S & \text{It is exact iff } f \text{ is flat.} \end{aligned}$$

$$f \text{ flat} \Rightarrow f^*: G(R) \rightarrow G(S).$$

Def. • An R -module M has Tor-dimension $\leq n$ if it has a resolution of length $\leq n$ by flat R -modules:

$$0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \quad F_i \text{ flat } R\text{-modules}$$

• $f: R \rightarrow S$ has Tor-dimension $\leq n$ if S has Tor-dim $\leq n$ as an R -module.

Rank. M has Tor-dim $\leq n \iff \text{Tor}_i^R(M, N) = 0 \quad \forall i > n \text{ and } N \in \text{Mod}_R$.

Example: • $\text{Tor-dim} \leq 0 \iff \text{flat}$

- If r is not a zero divisor in R , then

$$0 \rightarrow R \xrightarrow{r} R \rightarrow R/(r) \rightarrow 0$$

is exact, so $R/(r)$ has Tor-dim ≤ 1 .

- More generally, if (r_1, \dots, r_n) is a regular sequence in R , then $R \rightarrow R/(r_1, \dots, r_n)$ has Tor-dim $\leq n$.

Let $f: R \rightarrow S$ have Tor-dim $\leq n$. (R, S noetherian)

Let $\mathcal{C}_i \subset \text{Coh}(R)$ be the full subcategory of modules M such that

$$\text{Tor}_j^R(M, S) = 0 \quad \text{for } j > i.$$

So $\mathcal{C}_n = \text{Coh}(R)$, and $\mathcal{C}_0 \xrightarrow{f^*} \text{Coh}(S)$ is exact ($\text{Tor}_1^R(M, S) = 0$)

Moreover, $\mathcal{C}_{i-1} \subset \mathcal{C}_i$ satisfies the assumptions of the resolution theorem:

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0 : \quad \bullet \quad M \in \mathcal{C}_i, P \in \mathcal{C}_i \Rightarrow K \in \mathcal{C}_i$$

$$\bullet \quad M \in \mathcal{C}_i, P \in \text{Proj}(R) \Rightarrow K \in \mathcal{C}_{i-1}.$$

$$\Rightarrow K(\mathcal{C}_0) \xrightarrow{\sim} K(\text{Coh}(R))$$

$$f^* \downarrow$$

$$K(\text{Coh}(S)) \nearrow$$

Conclusion: for $f: R \rightarrow S$ of finite Tor-dimension, we get $f^*: G(R) \rightarrow G(S)$.

(can check: $g^* f^* \simeq (f \circ g)^*$...)

At the level of π_0 : $G_0(R) \xrightarrow{f^*} G_0(S)$

$$[M] \mapsto \sum_{i=0}^{\infty} (-1)^i [Tor_i^R(S, M)]$$

This also works for noetherian schemes: If $f: Y \rightarrow X$ is of finite Tor-dimension, then we have $f^*: G(X) \rightarrow G(Y)$.

Using similar tricks:

- For $f: Y \rightarrow X$ a proper morphism of noetherian schemes, we have

$$f_*: G(Y) \rightarrow G(X); \text{ on } \pi_0: f_*: G_0(Y) \rightarrow G_0(X)$$

$$[F] \mapsto \sum_{i=0}^{\infty} (-1)^i [R^i f_* F]$$

- For $f: Y \rightarrow X$ proper and of finite Tor-dimension,
(& X, Y are quasi-projective over a ring) then

$$f_*: K^{\text{perf}}(Y) \rightarrow K^{\text{perf}}(X)$$

$$\begin{array}{ccc} \text{Vect}(Y) & \xrightarrow{\quad \text{resolution theorem} \quad \text{applies} \quad} & \text{Vect}(X) \\ \cup & \swarrow & \downarrow \\ \{ \text{where } R^i f_* = 0 \text{ for } i > 0 \} & \xrightarrow{f_*} & \text{Vect}(X)_0 \end{array}$$

Example $X \xrightarrow{i} A_X^n \xrightarrow{p} X$: p is flat, i has Tor-dim $\leq n$

$$\begin{array}{c} R \leftarrow R[x_1, \dots, x_n] \rightarrow R \\ 0 \hookrightarrow x_i \end{array}$$

$$\rightsquigarrow G(X) \xrightarrow{p^*} G(A_X^n) \xrightarrow{i^*} G(X)$$

id

Warning: In this example, $i_*: \text{Coh}(R) \hookrightarrow \text{Coh}(R[x_1, \dots, x_n])$ does not land in the subcategory \mathcal{E}_0 on which i^* is exact. Hence, even though $i^* \circ i_*: \text{Coh}(R) \rightarrow \text{Coh}(R)$ is the identity, one cannot deduce that $i^* \circ i_*: G(R) \rightarrow G(R)$ is the identity.

In fact, $i_*: G(R) \rightarrow G(R[x_1, \dots, x_n])$ is null-homotopic.