

Theorem (Homotopy invariance of G-theory)

Let R be a noetherian comm. ring. Then the map $R \hookrightarrow R[t]$ induces a weak equivalence $G(R) \xrightarrow{\sim} G(R[t])$.
Hence, if R is regular, then $K(R) \xrightarrow{\sim} K(R[t])$.

- Remarks
- $K(R) \rightarrow K(R[t])$ is not an equivalence in general (e.g. $R = k[x]/(x^2)$)
 - If $R \neq 0$, $\text{Proj}(R)^\simeq \rightarrow \text{Proj}(R[t])^\simeq$ is not an equivalence.

$$\text{eg: } \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(R[t]) \setminus \text{GL}_2(R).$$

The Bass-Quillen conjecture states that $\pi_0(\text{Proj}(R)^\simeq) \xrightarrow{\sim} \pi_0(\text{Proj}(R[t])^\simeq)$

when R is regular noetherian. This is still open (known if R contains a field).

- Also true for schemes or non-commutative rings.

$\Leftrightarrow A \text{ noeth. and } A_{\geq 1} \text{ f.g. ideal}$

Lemma Let $A = \bigoplus_{n \geq 0} A_n$ be a $\overbrace{\text{graded noetherian comm. ring}}$. Suppose:

- 1) A is flat over A_0 .
- 2) A_0 has finite Tor-dimension over A (e.g. $A = A_0[E]$)

Then the functor $\text{Coh}(A_0) \rightarrow \text{Coh}_{\geq 0}^{\text{gr}}(A)$, $M \mapsto M \otimes_{A_0} A$
induces an isomorphism

$$\mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(A_0) \xrightarrow{\sim} K_*(\text{Coh}_{\geq 0}^{\text{gr}}(A))$$

↑ shift endomorphism.

Proof. $\mathcal{N} \subset \text{Coh}_{\geq 0}^{\text{gr}}(A)$ full subcategory where $\text{Tor}_i^A(A_0, -) = 0$ for $i > 0$.

so $\mathcal{N} \xrightarrow{- \otimes_{A_0} A} \text{Coh}(A_0)$ is exact.

By the resolution theorem, $K(\mathcal{N}) \xrightarrow{\sim} K(\text{Coh}_{\geq 0}^{\text{gr}}(A))$.

Recall: $F_n M \subset M$ sub- A -module generated in degrees $\leq n$

Let $N_n \subset \mathcal{N}$ be the subcategory where $F_n M = M$.

So $\mathcal{N} = \text{colim}_{n \rightarrow \infty} M_n$.

Claim: $K_*(N_n) \simeq \bigoplus_{k=0}^n G_*(A_0)$

$$\begin{array}{ccc} \prod_{k=0}^n \text{Coh}(A_0) & \xrightleftharpoons[\beta]{\alpha} & N_n \\ & & \downarrow \text{shift up by } k. \\ (M_k)_{k=0}^n & \longmapsto & \bigoplus_{k=0}^n A(-k) \otimes_{A_0} M_k \\ \left((M \otimes_{A_0})_k \right)_{k=0}^n & \longleftarrow & M \end{array}$$

$\Rightarrow \beta \circ \alpha \simeq \text{id}$, $\alpha \circ \beta$ is the associated graded of the filtration $\{F_k\}$ of id_{N_n} .

$\Rightarrow K(\alpha \circ \beta) \simeq K(\text{id}_{N_n})$ by the additivity theorem. \square

This proves the claim.

Proof of homotopy invariance ($G(R) \simeq G(R[t])$)

Let $A = \bigoplus_{n \geq 0} (R[t]_{\leq n})_{\mathbb{Z}^n} \subset R[t, z]$ subring.

- $A/zA \simeq R[t]$
 $[t^n z^n] \longleftrightarrow t^n$
- A is flat over R
- $z \in A$ is not a zero divisor $\Rightarrow A/zA \simeq R[t]$ has Tor-dim ≤ 1 over A .

$\Rightarrow R[t]/(t) \simeq R$ has Tor-dim ≤ 2 over A .

- A is noetherian: $R[u, v] \rightarrow A$, $u \mapsto z$, $v \mapsto bz$.

By the lemma:

$$\mathbb{Z}[x] \otimes G_i(R) \xrightarrow{\sim} K_i(\text{Coh}_{\geq 0}^{gr}(A))$$

$$\mathbb{Z}[x] \otimes G_i(R) \xrightarrow{\sim} K_i(\text{Coh}_{\geq 0}^{gr}(R[t]))$$

Let $B \subset \text{Coh}_{\geq 0}^{gr}(A)$ be the Serre subcategory where z is nilpotent.

Then $\text{Coh}_{\geq 0}^{gr}(A)/B \xrightarrow{\sim} \text{Coh}(R[t])$, $M \mapsto M/(z-1)M$

$\text{Coh}_{\geq 0}^{\text{fr}}(R[t]) \hookrightarrow \mathcal{B}$ subcategory where t is zero

By devissage, $K(\text{Coh}_{\geq 0}^{\text{fr}}(R[t])) \xrightarrow{\sim} K(\mathcal{B})$,

By localization:

$$\begin{array}{ccccccc} \cdots & \rightarrow G_{*+1}(R[t]) & \rightarrow K_*(\text{Coh}_{\geq 0}^{\text{fr}}(R[t])) & \rightarrow K_*(\text{Coh}_{\geq 0}^{\text{fr}}(A)) & \rightarrow G_*(R[t]) & \rightarrow \cdots \\ & & \uparrow \cong & & \downarrow \cong & & \\ & & \mathbb{Z}(x) \otimes G_*(R) & \xrightarrow{1-x} & \mathbb{Z}(x) \otimes G_*(R) & & \\ & & & \nearrow & & & \\ 0 \rightarrow M_R \otimes A(-) & \xrightarrow{\cong} & M_R \otimes A & \rightarrow M_R \otimes R[t] & \rightarrow 0 & & (\text{use additivity theorem}) \end{array}$$

\Rightarrow the LES consists of SES

$$0 \rightarrow \mathbb{Z}(x) \otimes G_*(R) \xrightarrow{1-x} \mathbb{Z}(x) \otimes G_*(R) \rightarrow G_*(R[t]) \rightarrow 0$$

$$\Rightarrow G_i(R[t]) = \text{coker}(1-x) \cong G_i(R). \quad \square$$

Corollary If R is a regular domain, then

$$G_*(R[t, t^{-1}]) \cong G_*(R) \oplus G_{*-1}(R)$$

Pf. Apply localization to $\{0\} \hookrightarrow A_R^1 \hookleftarrow A_R^1 \setminus \{0\} = \text{Spec } R[t^{\pm 1}]$

$$\begin{array}{ccccccc} \rightsquigarrow & G_i(R) & \rightarrow & G_i(R[t]) & \rightarrow & G_i(R[t^{\pm 1}]) & \rightarrow G_{i-1}(R) \\ & & \uparrow \cong & & \searrow \text{induced by} & & \\ & & G_i(R) & & R[t^{\pm 1}] \rightarrow R & & \text{Tor-dim} \leq 1. \\ & & & & t \mapsto 1 & & \end{array}$$

\Rightarrow we have split SES

$$0 \rightarrow G_i(R[t]) \rightarrow G_i(R[t^{\pm 1}]) \rightarrow G_{i-1}(R) \rightarrow 0$$

\square

Remark: If R is regular, we get

$$K_*(R[t^{\pm 1}]) \cong K_*(R) \oplus K_{*-1}(R)$$

In fact, this holds for all R for $i \geq 1$. This is the so-called "Bass fundamental theorem". One uses this to define negative K -groups (which are zero if R is regular).

Corollary Let X be a noetherian scheme and $f: E \rightarrow X$ a flat morphism whose fibers are affine spaces (E noeth.).

(i.e. $E_x \cap \text{Spec } k(x) \cong \mathbb{A}_{k(x)}^n$ over $k(x)$ for all $x \in X$).

(e.g. $E \rightarrow X$ is a vector bundle).

Then $f^*: G(X) \rightarrow G(E)$ is a weak equivalence.

Pf. If $Z \subset X$ closed subscheme with open complement $U \subset X$, we have homotopy fiber sequences

$$\begin{array}{ccc} G(Z) & \rightarrow & G(X) \rightarrow G(U) \\ \downarrow & \downarrow f^* & \downarrow \\ G(E_Z) & \rightarrow & G(E) \rightarrow G(E_U). \end{array}$$

We say that a subscheme $Y \subset X$ is good if $G(Y) \xrightarrow{\sim} G(E_Y)$.

Note: \emptyset is good. By noetherian induction, we can assume that Z is good for every closed $Z \subset X$ with $X \cdot Z \neq \emptyset$.

If X is not irreducible: $X = Z_1 \cup Z_2$, Z_1, Z_2 and $Z_1 \cap Z_2$ are good.

$$Z_2, Z_1 \cap Z_2 \text{ good} \xrightarrow{\text{Induction}} \underbrace{Z_2 \setminus (Z_1 \cap Z_2)}_{X \setminus Z_1} \text{ is good}$$

$$Z_1, X \cdot Z_1 \text{ good} \Rightarrow X \text{ good}$$

So we can assume X irreducible, with generic point $\eta \in X$.

$$\eta = \bigcap_{\substack{Z \subset X \\ \text{closed}}} X \cdot Z \Rightarrow \text{Coh}(\eta) \cong \varprojlim_{\substack{Z \not\subset X \\ \text{closed}}} \text{Coh}(X \cdot Z)$$

$$E_\eta = \bigcap_{\substack{Z \not\subset X \\ \text{closed}}} E_{X \cdot Z} \Rightarrow \text{Coh}(E_\eta) = \varprojlim_{\substack{Z \not\subset X \\ \text{closed}}} \text{Coh}(E_{X \cdot Z})$$

$$\text{Spec}(k(\eta)[t_1, \dots, t_n])$$

(this uses: if $R = \varprojlim R_2$, then $\text{Mod}^{\text{fp}}(R) \cong \varprojlim \text{Mod}^{\text{fp}}(R_2)$.)

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & \underset{\substack{z \in X \\ z \neq x}}{\operatorname{colim}} G_*(\mathbb{H}) & \rightarrow & G_*(X) & \rightarrow & \underset{\substack{z \in X \\ z \neq x}}{\operatorname{colim}} G_*(X, z) \cong G_*(\eta) \rightarrow \cdots \\
 & & \downarrow & f^* \downarrow & & \downarrow & \\
 \cdots & \rightarrow & \underset{\substack{z \\ z \in X}}{\operatorname{colim}} G_*(E_z) & \rightarrow & G_*(E) & \rightarrow & \underset{\substack{z \\ z \in X}}{\operatorname{colim}} G_*(E_{X, z}) \cong G_*(A_\eta^n) \rightarrow \cdots
 \end{array}$$

By homotopy invariance, $G_*(\eta) \xrightarrow{\sim} G_*(A_\eta^n)$.

By induction hypothesis, the left vertical map is an isomorphism.

\Rightarrow the middle map is an isomorphism (by 5-lemma
(and $G_0(X) \rightarrow G_0(\eta)$)) \blacksquare