

The projective bundle formula

X scheme, \mathcal{E} quasi-coherent sheaf on X

$$\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}_{\mathcal{O}_X}(\mathcal{E})) \quad , \quad \pi: \mathbb{P}(\mathcal{E}) \rightarrow X$$

eg: $\mathbb{P}(\mathcal{O}_X^{n+1}) = \mathbb{P}_X^n$

for $x \in X$, $\mathbb{P}(\mathcal{E})_x \cong \mathbb{P}(\mathcal{E}_{K(x)})$

\exists canonical line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$, $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, $n \in \mathbb{Z}$

For X noetherian, \mathcal{E} coherent: $-\infty < n < \infty$: $\text{Coh}(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} \text{Coh}(\mathbb{P}(\mathcal{E}))$

($\Rightarrow \mathbb{P}(\mathcal{E})$ noetherian)

induces $(-)(n): G_*(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} G_*(\mathbb{P}(\mathcal{E}))$

Suppose $X = \text{Spec } R$. Then

$$\text{Coh}(\mathbb{P}(\mathcal{E})) \cong \text{Coh}_{\geq 0}^{\text{gr}}(\text{Sym}_R \mathcal{E}) / \text{Coh}_{> 0, < \infty}^{\text{gr}}(\text{Sym}_R \mathcal{E})$$

Theorem (Projective bundle formula)

X noetherian scheme, \mathcal{E} finite locally free \mathcal{O}_X -module of rank $n+1$.

$\pi: \mathbb{P}(\mathcal{E}) \rightarrow X$. Then

$$G(\mathbb{P}(\mathcal{E})) \cong G(X)^{n+1}$$

$$\sum_{i=0}^n \pi^*(\alpha_i)(-i) \longleftarrow (d_i)_{i=0}^n$$

Proof sketch

As in the previous corollary, we can reduce to the case $X = \text{Spec } k$,

k a field, $\mathcal{E} = E$ vector space over k of dim $n+1$.

By localization:

$$K(\text{Coh}_{> 0, < \infty}^{\text{gr}}(\text{Sym}_k E)) \rightarrow K(\text{Coh}_{\geq 0}^{\text{gr}}(\text{Sym}_k E)) \rightarrow G(\mathbb{P}(\mathcal{E}))$$

By devissage,

$$K(\text{Coh}_{> 0, < \infty}^{\text{gr}}(\text{Sym}_k E)) \cong K(\text{Coh}_{\geq 0}^{\text{gr}}(k))$$

\uparrow
 filter M by $(\text{Sym}_k^i E)^i M$
 ($= 0$ if $i > 0$)

By previous lemma:

$$\begin{array}{ccc} K_* (\text{Coh}_{\geq 0}^{\text{gr}}(k)) & \longrightarrow & K_* (\text{Coh}_{\geq 0}^{\text{gr}}(\text{Sym}_k E)) \\ \sim \uparrow & & \sim \uparrow \\ \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) & \xrightarrow{h} & \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) \end{array}$$

What is h ?

Use Koszul resolutions: $M \rightarrow R$ R -linear map
 \rightsquigarrow Koszul complex $\cdots \rightarrow \Lambda_R^i M \xrightarrow{d} \Lambda_R^{i-1} M \rightarrow \cdots \rightarrow M \rightarrow R$
 For $R^n \rightarrow R$, Koszul complex is exact if $R^n \rightarrow R$ defines a regular sequence.
 • If $M = R \otimes_{\mathbb{Z}} N$, then $\Lambda_R^i M \cong R \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}^i N$.

Apply this to $\text{Sym}_k(E) \otimes_{\mathbb{Z}} E(-i) \rightarrow \text{Sym}_k(E) (= k \oplus E(-1) \oplus \cdots)$

\rightsquigarrow we get an exact sequence:

$$0 \rightarrow \text{Sym}_k(E) \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}^{n+1} E(-n-1) \rightarrow \cdots \rightarrow \text{Sym}_k(E) \otimes_{\mathbb{Z}} E(-1) \rightarrow \text{Sym}_k(E)$$

\rightsquigarrow get an exact sequence of functors

$$\text{Coh}_{\geq 0}^{\text{gr}}(k) \rightarrow \text{Coh}_{\geq 0}^{\text{gr}}(\text{Sym}_k E)$$

$$\begin{array}{c} \downarrow \\ R \\ \downarrow \\ 0 \end{array}$$

by turning with the terms of this sequence. By the additivity theorem, we deduce that:

$$h = \sum_{i=0}^{n+1} (-1)^i x^i (\Delta^i \xi \otimes_{\mathbb{Z}} -)_* : \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) \rightarrow \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k).$$

$$\cdots \rightarrow K_* (\text{Coh}_{\geq 0}^{\text{gr}}(k)) \rightarrow K_* (\text{Coh}_{\geq 0}^{\text{gr}}(\text{Sym}_k E)) \rightarrow G_*(P(\mathcal{E})) \rightarrow \cdots$$

$$\begin{array}{ccc} \sim \uparrow & & \sim \uparrow \\ \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) & \xrightarrow{h} & \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) \end{array}$$

Since $h = \text{id} + x(\cdots)$, h is injective, so we get SES:

$$0 \rightarrow \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) \xrightarrow{h} \mathbb{Z}[x] \otimes_{\mathbb{Z}} G_*(k) \rightarrow G_*(P(\mathcal{E})) \rightarrow 0$$

\nwarrow coefficient of x^{n+1} is invertible
 $\sum_{i=0}^n \alpha_i x^i$
 \uparrow
 $G_*(k)^{n+1}$
 \searrow
 $(\alpha_i)_{i=0}^n$
 $\rightarrow \sum_{i=0}^n \pi^*(\alpha_i)(-i)$

□

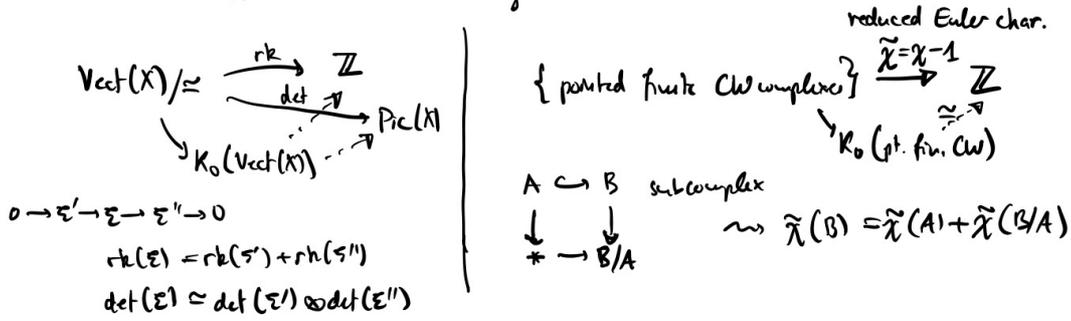
Remark In topology, we have the analogous formula:

$$H^*(P(V)) \cong \bigoplus_{i=0}^n H^{*-2i}(X)$$

when $V \rightarrow X$ is a complex vector bundle of rank $n+1$.

Waldhausen S.-construction

Motivation: 1) non-additive categories



2) functoriality of K-theory / G-theory of schemes.

Recall: • $f: Y \rightarrow X$ finite Tor dim $\Rightarrow f^*: G_0(X) \rightarrow G_0(Y)$
 $[M] \mapsto \sum (-1)^i [\text{Tor}_i^{O_Y}(O_X, M)]$

• $f: Y \rightarrow X$ proper $\Rightarrow f_*: G_0(Y) \rightarrow G_0(X)$, $[M] \mapsto \sum (-1)^i [Rf_* M]$

This suggests that $G(X)$ should depend on $D_{\text{coh}}(X)$, where

we have $Rf_*: D_{\text{coh}}(Y) \rightarrow D_{\text{coh}}(X)$

$Rf_*: D_{\text{coh}}(Y) \rightarrow D_{\text{coh}}(X)$

Waldhausen categories

Def. A Waldhausen category is a category \mathcal{C} with two classes of morphisms, cofibrations and weak equivalences, such that:

- 1) cofibrations and weak equivalences are closed under composition and contain isomorphisms.

"category with cofibrations"

- 2) \mathcal{C} has a zero object 0
- 3) $\forall A \in \mathcal{C}, 0 \rightarrow A$ is a cofibration.
- 4) Cofibrations are preserved by cobase change:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & p_0 & \downarrow \\ C & \longrightarrow & B \amalg_A C \end{array}$$

- 5) Given

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

then the induced map $B \amalg_A C \rightarrow B' \amalg_{A'} C'$ is a weak equivalence.

Examples

- 1) \mathcal{C} = exact category with cofibrations = colim mono
weak equiv = isomorphisms
- 2) (finite) pointed CW complexes with : cofibrations = subcomplex inclusions
weak equiv = htpy equiv.
- 3) (bounded) chain complexes in an abelian category \mathcal{A}
with cofibrations = degreewise split mono
w.e. = quasi-iso.

Def. $K_0(\mathcal{C})$ = abelian group generated by $[A], A \in \mathcal{C}$, with relations:

$$1) A \xrightarrow{\sim} B \Rightarrow [A] = [B]$$

$$2) \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & p_0 & \downarrow \\ 0 & \longrightarrow & C \end{array} \Rightarrow [B] = [A] + [C].$$

Examples

- 1) For \mathcal{C} exact, $K_0(\mathcal{C})$ is the same as before.
- 2) $K_0(\text{pt. finite CW complexes}) \cong \mathbb{Z}$ via the Euler characteristic
- 3) $K_0(\text{bounded chain complexes in } \mathcal{A}) \cong K_0(\mathcal{A})$

Want a space $K(\mathcal{C})$ such that $\pi_0 K(\mathcal{C}) = K_0(\mathcal{C})$.