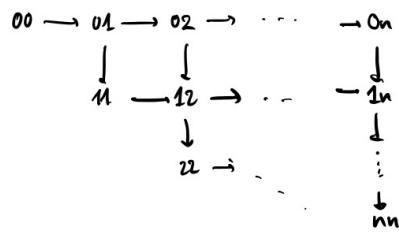


Notation: $W\mathcal{C} \subset \mathcal{C}$ subcategory of weak equivalences

The S_n -construction

let $\text{Ar}[n] = \{(i,j) \mid 0 \leq i \leq j \leq n\}$ poset with $(i,j) \leq (k,l)$
iff $i \leq k$ and $j \leq l$.



Note $\text{Ar}[n] = \text{Fun}([1], [n])$ where $[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$.

$\Rightarrow [n] \mapsto \text{Ar}[n]$ is a functor $\Delta \rightarrow \text{Cat}$

Let \mathcal{C} be a Waldhausen category,

Let $S_n \mathcal{C} \subset \text{Fun}(\text{Ar}[n], \mathcal{C})$ be the full subcategory with objects

$$(X_{ij}) \text{ s.t. } \begin{aligned} & \bullet \forall i, X_{ii} \simeq 0 \\ & \bullet \forall i \leq j \leq k, X_{ij} \rightarrowtail X_{ik} \\ & \quad \downarrow \quad \text{pr}_0 \quad \downarrow \\ & \quad 0 \simeq X_{jj} \rightarrowtail X_{jk} \end{aligned}$$

$$S_0 \mathcal{C} = \{0\}$$

$$S_1 \mathcal{C} \simeq \mathcal{C}$$

$$S_2 \mathcal{C} \simeq \left\{ \begin{array}{c} A \rightarrowtail B \\ \downarrow \text{pr}_0 \downarrow \\ 0 \rightarrowtail C \end{array} \right\} \simeq \{\text{cofibrations in } \mathcal{C}\} \subset \text{Fun}([2], \mathcal{C})$$

$$\underline{\text{Remark}} \quad S_n \mathcal{C} \simeq \{A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n\} \subset \text{Fun}([n-1], \mathcal{C}).$$

Given $\varphi: [m] \rightarrow [n]$, in Δ , $\text{Fun}(\text{Ar}[n], \mathcal{C}) \supset S_n \mathcal{C}$

$$\varphi^* \downarrow \quad \quad \quad \downarrow \exists \varphi^* \\ \text{Fun}(\text{Ar}[m], \mathcal{C}) \supset S_m \mathcal{C}$$

$\Rightarrow [n] \mapsto S_n \mathcal{C}$ is a functor $\Delta^{\text{op}} \rightarrow \text{Cat}$.

$$\begin{array}{ccccc}
 S_0\mathcal{C} & \xleftarrow{\quad} & S_1\mathcal{C} & \xleftarrow{\quad} & S_2\mathcal{C} \dots \\
 \parallel & & \parallel & & \parallel \\
 \{0\} & & \mathcal{C} & & \{A \rightarrow B\} \\
 \circ \xrightarrow{s_0} & & \circ & & \\
 & & & & A \xleftarrow{d_0} \\
 & & & & B \xleftarrow{d_1} A \rightarrow B \\
 & & & & B/A \xleftarrow{d_2}
 \end{array}$$

Definition $K(\mathcal{C}) = \Omega_0 \left[[n] \mapsto |N(wS_n\mathcal{C})| \right]$

Easy: $\pi_0 K(\mathcal{C}) = \pi_1 \left[[n] \mapsto |N(wS_n\mathcal{C})| \right] \simeq K_0(\mathcal{C}).$

Remarks.

- If w.e. = isos, we can replace $wS_n\mathcal{C}$ by $\text{ob}(S_n\mathcal{C})$ and get an equivalent space.
- If \mathcal{C} is an exact category, then this recovers Quillen's K-theory. In fact, $N(Q\mathcal{C})$ is the edgewise subdivision of $\text{ob}(S_n\mathcal{C})$.

Edgewise subdivision: $\Delta^e: \Delta \rightarrow \Delta$

$$d: \Delta \rightarrow \Delta, I \mapsto I^{\text{op}} * I, \text{ e.g. } \begin{matrix} 0 < 1 < \dots < n \\ \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\ n' < \dots < q' < 0' < o' < \dots < n \end{matrix}$$

$$[n] \mapsto [2n+1]$$

If $X \in \text{sset}$, its edgewise subdivision is $X \circ d$.

$|X|$ is homeomorphic to $|X \circ d|$.

Theorem (Additivity)

If \mathcal{C} is a Waldhausen category, then

$$K(S_2\mathcal{C}) \rightarrow K(\mathcal{C}) \times K(\mathcal{C})$$

induced by $\begin{array}{c} A \rightarrow B \\ \downarrow p_B \downarrow \\ 0 \rightarrow \mathcal{C} \end{array} \mapsto (A, C)$

is a weak equivalence.

(see Waldhausen, "Algebraic K-theory of spaces", §1)

A-theory $X \in \text{Top}$.

$$\mathcal{R}(X) = (\text{Top}/X)_* = \left\{ \begin{array}{c} Y \\ f \downarrow X \end{array} \right\}_s, f \circ s = \text{id}_X \right\}$$

$\mathcal{R}_{fd}(X) \subset \mathcal{R}(X)$ full subcategory containing (Y, f, s) where $s(X) \subset Y$ is a finite relative CW complex, and closed under homotopy equivalences and retracts.

- cofibrations : subcomplex inclusions
- w.c. = w.c.

Def $A(X) = K(\mathcal{R}_{fd}(X))$.

so $A(*) = K(\text{pt. finitely dominated CW complexes})$

If X is connected and "nice", $\pi_0 A(X) \cong K_0(\mathbb{Z}[\pi_1 X])$ and $\pi_1 A(X) \cong K_1(\mathbb{Z}[\pi_1 X])$ (but $\pi_n \neq K_n$ for $n \geq 2$)

Thm (Waldhausen)

$$A(*) \cong (\text{Fin}^{\infty})^{\partial} \times \text{Wh}(*) \text{ as } E_{\infty}\text{-groups}$$

more generally: $A(X) \cong \underbrace{\Sigma^n \Sigma_+^\infty X}_{\text{column}} \times \text{Wh}(X)$

K-theory of schemes (Thomason - Trobaugh)

Def. A perfect complex on a scheme X is a chain complex of \mathcal{O}_X -modules locally quasi-isomorphic to a bounded complex of vector bundles.

Fact: If $X = \text{Spec } R$ is affine (more generally if X is quasi-projective with affine depth and has the resolution property) then every perfect complex on X is globally q-i. to a bounded complex of vector bundles.

$\text{Perf}(X) = \text{category of perfect complexes} \subset \text{Ch}(\mathcal{O}_X\text{-mod})$

- cofibrations : degreewise split monomorphisms
- weak equivalences : quasi-isomorphisms

$\rightarrow \text{Perf}(X)$ is a Waldhausen category.

Also, if $Z \subset X$ closed subset, $\text{Perf}_Z(X) \subset \text{Perf}(X)$, full subcategory of complexes P s.t. $P|_{X-Z} \simeq_0 0$, is a Waldhausen category.

Def $K(X) = K(\text{Perf}(X))$, $K_Z(X) = K(\text{Perf}_Z(X))$.

If X is quasi-projective over a ring,

or if X is noetherian regular separated, then $K(X) \simeq K^{\text{naive}}(X)$.

Theorem (Thomason) $K: \text{Sch}_{\text{quasi}}^{\text{op}} \rightarrow E_{\infty}\text{-groups}$ satisfies:

- Mayer-Vietoris : $K(U_1 \cup U_2) \rightarrow K(U_2)$

$$\begin{array}{ccc} & \downarrow & \downarrow \\ K(U_1) & \rightarrow & K(U_1 \cap U_2) \end{array} \quad \text{homotopy pullback square.}$$

- projective bundle formulae

$$K(P_X^{\mathbb{P}^n}) \simeq \prod_{i=0}^n K(X).$$

- Bass fundamental theorem :

$$\begin{array}{c} S^2 K(\mathbb{G}_m \times X) \simeq S^2 K(X) \times K(X) \\ \uparrow \\ \text{Spec } \mathbb{Z}[t^{\pm 1}] \end{array}$$

G-theory X noetherian.

$D\text{Coh}(X) \subset \text{Ch}(\mathcal{O}_X\text{-mod})$: complexes M_* s.t. $\bigoplus_{i \in \mathbb{Z}} H_i(M_*)$ coherent.

- cofibrations = degreewise split monos
- w.e. = quasi-isomorphisms

Thm: $G(X) \simeq K(D\text{Coh}(X))$.

In general, $\text{Perf}(X) \subset D\text{Coh}(X) \Rightarrow$ canonical map $K(X) \rightarrow G(X)$.

X is regular iff $\text{Perf}(X) = D\text{Coh}(X)$, in which case $K(X) \simeq G(X)$.