

Suslin's rigidity theorem

K-theory with finite coefficients

Idea:

$$K(\mathcal{C}) \xrightarrow{\cdot^n} K(\mathcal{C}) \longrightarrow K(\mathcal{C})/n$$

\cong homotopy colimit in \mathbb{E}_∞ -groups.

$$\text{We define: } (K/n)_*(\mathcal{C}) = K_*(\mathcal{C})/n$$

$$(K/n)_i(\mathcal{C}) = \pi_{i-1}(\text{hofib}(K(\mathcal{C}) \xrightarrow{\cdot^n} K(\mathcal{C})))$$

Warning: $(K/n)_*(\mathcal{C})$ is not a \mathbb{Z}/n -module in general,
but it is a \mathbb{Z}/n^2 -module.

Theorem (Suslin rigidity)

Let $k \subset k'$ be an extension of algebraically closed fields, $n \geq 1$. Then

$$(K_n)_*(k) \xrightarrow{\sim} (K/n)_*(k').$$

This is a key ingredient in the following:

Theorem (Suslin)

If $k = \bar{k}$, $n \geq 1$.

$$1) \quad \text{If } \sqrt[n]{a} \in k, \text{ then } (K/n)_*(k) = \begin{cases} \widetilde{\mathbb{Z}/n} & \text{non-canonically} \\ \mu_n^{\otimes r} & \text{if } n=2r \\ 0 & \text{otherwise} \end{cases}$$

$$2) \quad \text{If } \text{char}(k)=p>0, \text{ then } (K/p^n)_*(k) = \begin{cases} \mathbb{Z}/p^n & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

⚠ From now on: $K = k/n$, so $K_*(X)$ is n -power torsion.

Proof of rigidity k perfect $\Rightarrow k' = \varprojlim_{\substack{k \subset A \subset k' \\ A \text{ smooth/k}}} A$ (filtered)

$$\Rightarrow K_*(k') = \varprojlim_A K_*(A).$$

For any A , \exists k -algebra map $A \rightarrow k$ (Nullstellensatz)

$\Rightarrow K_*(k) \rightarrow K_*(A)$ is split injective.

$\Rightarrow K_*(k) \rightarrow K_*(k')$ is injective.

For surjectivity, it suffices to show that the image of $K_*(A) \rightarrow K_*(k')$ is contained in $K_*(k) \hookrightarrow K_*(k')$, for all A .

This follows from

$$\begin{array}{ccc} \text{Spec } A \otimes k' & \xrightarrow{\quad} & \text{Spec } A \\ \text{Spec } k' \downarrow & \nearrow x_0 \quad \nearrow x_1 & \downarrow \\ & & \text{Spec } k \end{array} \quad \underline{\text{Claim}}: \quad x_0^* = x_1^*: K_*(A) \rightarrow K_*(k').$$

Replacing A by $A \otimes k'$, we can assume $k = k'$.

Theorem Let X be smooth connected of finite type over $k = \bar{k}$, and $x_0, x_1: \text{Spec } k \rightarrow X$, then $x_0^* = x_1^*: (K/k)_*(X) \rightarrow (K/k)_*(k)$.

Proof. X smooth connected $\Rightarrow \exists$ smooth ^{connected} curve $C \subset X$ containing x_0, x_1 .

WLOG, $X = C$ is a curve.

Case $C = \mathbb{P}_k^1$: x_0, x_1 are A^1 -homotopic $\Rightarrow x_0^* = x_1^*$ by A^1 -invariance

$$\begin{array}{ccc} \text{Spec } k & & \\ \text{of } A^1 & \xrightarrow{\exists} & \mathbb{P}^1 \\ \uparrow & & \\ \text{Spec } k & \xrightarrow{x_1} & x_1 \end{array}$$

(could also use the projective bundle formula).

$$\text{General case} \quad \text{Div}(C) = \mathbb{Z}[C(k)] \xrightarrow{\text{deg}} \mathbb{Z}$$

$$\text{Div}^0(C) = \ker(\text{deg}).$$

$$q_C: \text{Div}(C) \otimes K_*(C) \longrightarrow K_*(k)$$

$$\sum n_i x_i \otimes \alpha \longmapsto \sum n_i x_i^*(\alpha)$$

Goal: q_C vanishes on $\text{Div}^0(C)$

Step 1 Birational reduction. Let $x \in C(k)$, let E be the function field of C .

$$\begin{array}{ccc} K_*(C) & \xrightarrow{x^*} & K_*(k) \\ \downarrow & \nearrow \exists x! & \\ K_*(E) & & \end{array}$$

Construction of $x!$ Let $R = \mathcal{O}_{C,x}$, dvr

$$\begin{array}{c}
 \text{as left sequence } K(k) \rightarrow K(R) \rightarrow K(E) \\
 \left(\begin{array}{c} \oplus, \otimes \\ \rightsquigarrow K_*(X) \\ \text{a graded ring} \end{array} \right) \\
 \begin{array}{ccccc}
 & K_1(E) & & & \\
 \widetilde{E^*} \otimes K_*(E) & \xrightarrow{\quad} & K_{*-1}(E) & \xrightarrow{\partial} & K_*(k) \\
 \uparrow & & \uparrow \text{(forget)} & & \uparrow \\
 R^* \otimes K_*(E) & \longrightarrow & \underbrace{k^* \otimes K_*(E)}_{\substack{\text{divisible} \\ (x^n - a = 0 \\ \text{has a solution})}} & \xrightarrow{\partial} & k^* \otimes K_{*-1}(k) \\
 & & \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{0} & & \\
 & & \simeq \mathbb{Z} & & \\
 \Rightarrow \quad \widetilde{E^*/R^*} \otimes K_*(E) & \dashrightarrow & K_*(k) & & \\
 & \parallel & & \nearrow x! &
 \end{array}
 \end{array}$$

so \bar{q}_C factors through:

$$\bar{q}_C: \text{Div}(C) \otimes K_*(E) \rightarrow K_*(k)$$

$$\sum n_i x_i \otimes \alpha \mapsto \sum n_i x_i^!(\alpha)$$

Claim: \bar{q}_C vanishes on $\text{Div}^0(C)$

wLOG C is proper (thus doesn't change E)

Step 2 \bar{q}_C factors through $\text{Pic}(C) = \text{Div}(C)/\text{rational equivalence}$

$$\begin{array}{ccc}
 E^* \longrightarrow \text{Div}(C) & & \text{Pic}(C) \text{ is the cokernel.} \\
 f \longmapsto (f) = \sum_{x \in C(k)} \text{ord}_x(f) \cdot x & & \\
 & \parallel & \\
 \zeta_0 \hookrightarrow C \hookrightarrow \zeta_\infty & \sum_{x \in \zeta_0} \dim_k(\mathcal{O}_{C,x}) \cdot x - \sum_{x \in \zeta_\infty} \dim_k(\mathcal{O}_{C,x}) \cdot x. &
 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \hookrightarrow \mathbb{P}_k^1 \hookrightarrow \infty$$

$$\rightsquigarrow k(t) \subset E \text{ finite field extension} \Rightarrow \text{tr}_{E/k(t)}: K(E) \rightarrow K(k(t)).$$

Using basic properties of traces, one can compute:

$$\bar{q}_C((f) \otimes \alpha) = \bar{q}_{\mathbb{P}^1}((0-\infty) \otimes \text{tr}_{E/k(t)}(\alpha)) \stackrel{\text{P}^1\text{-coker.}}{=} 0$$

\Rightarrow we get $\text{Pic}(C) \otimes K_*(E) \longrightarrow K_*(k)$.

$$C \text{ proper} \Rightarrow E^* \longrightarrow \text{Div}^0(C) \subset \text{Div}(C)$$

$$\text{Pic}^0(C) = \text{coker}(E^* \longrightarrow \text{Div}^0(C)).$$

k -algebraically closed $\Rightarrow \text{Pic}^0(C)$ is divisible.

- $\text{Pic}^0(C)$ is the group of rational points of an abelian variety $/k$
- If abelian variety $A/k=\bar{k}$, $A(\bar{k})$ is divisible
(more generally, over any field, $n: A \rightarrow A$ is surjective).

$\text{Pic}^0(C)$ divisible, $K_*(E)$ torsion $\Rightarrow \text{Pic}^0(C) \otimes K_*(E) = 0$.

Hence \bar{q}_C vanishes on Div^0 . □