

The determinant  $K_0(R) \rightarrow \text{Pic}(R)$  ( $R$  commutative)

$\text{Pic}(R)$  = iso classes of fg-proj  $R$ -modules of rank 1.

e.g.  $\text{Pic}(\mathcal{O}_K) = \text{Cl}(K)$  ideal class group (for  $K$  a number field).

Def.  $M$  an  $R$ -module and  $n \geq 0$ . The symmetric group  $\Sigma_n$  acts on  $M^{\otimes n}$  by permuting factors.

1) The  $n$ -th symmetric power of  $M$  is

$$\text{Sym}^n(M) = (M^{\otimes n})_{\Sigma_n} = M^{\otimes n} / \langle m_1 \otimes \dots \otimes m_n - m_{\sigma(1)} \otimes \dots \otimes m_{\sigma(n)} \mid \sigma \in \Sigma_n \rangle$$

2) The  $n$ -th divided power of  $M$  is

$$(M \text{ flat}) \quad T^n(M) = (M^{\otimes n})^{\Sigma_n} \subset M^{\otimes n} \quad (\text{invariants})$$

3) The  $n$ -th exterior power of  $M$  is

$$\Lambda^n(M) = M^{\otimes n} / \langle m_1 \otimes \dots \otimes m_n \mid m_i = m_j \text{ for some } i \neq j \rangle$$

Notation:  $m_1 \wedge \dots \wedge m_n$  is the image of  $m_1 \otimes \dots \otimes m_n$  in  $\Lambda^n M$ .

Remark there are canonical isomorphisms  $T^n(M) \cong (\text{Sym}^n(M))^{\vee}$

if  $M$  is fg-projective.  $\Lambda^n(M) \cong (\Lambda^n(M^{\vee}))^{\vee}$

Lemma. If  $M$  is free of finite rank with basis  $e_1, \dots, e_d$ , then:

1)  $\text{Sym}^n(M)$  is free of rank  $\binom{n+d-1}{n}$  with basis  $e_1^{\alpha_1} \otimes \dots \otimes e_d^{\alpha_d}$  with  $\sum \alpha_i = n$

2)  $T^n(M)$  is free of rank  $\binom{n+d-1}{n}$

3)  $\Lambda^n(M)$  is free of rank  $\binom{d}{n}$  [ $= 0$  if  $n > d$ ] with basis

Proof 3) Define  $M^n \rightarrow M^{\otimes n}$

$$(x_1, \dots, x_n) \mapsto \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}.$$

is multilinear and alternating  $\Rightarrow$  induces  $\alpha: \Lambda^n(M) \rightarrow M^{\otimes n}$ .

$$\alpha \left( \sum_I c_I (e_{i_1} \wedge \dots \wedge e_{i_n}) \right) = \sum_I \sum_{\sigma} c_I \text{sign}(\sigma) \underbrace{(e_{\sigma(i_1)} \otimes \dots \otimes e_{\sigma(i_n)})}_{\text{Linearly indep. in } M^{\otimes n}}$$

$e_{i_1} \wedge \dots \wedge e_{i_n}$  are lin. indep.

Corollary If  $M$  is fpf of rank  $d$ , then  $\text{Sym}^n(M)$ ,  $T^n(M)$ ,  $\Lambda^n(M)$  are fpf of rank as above.

Pf:  $M$  is a retract of  $R^m$  for some  $m \Rightarrow \text{Sym}^n(M)$  is a retract of  $\text{Sym}^n(R^m)$ .

$\Rightarrow \text{Sym}^r(M), \text{Tr}^n(M), \Lambda^k(M)$  are fd. proj of finite rank.

To compute the rank, we can  $\xrightarrow{(-1)^{\otimes k} R} K(R)$ . □

Def. Let  $M \in \text{Proj}(R)$ . The determinant of  $M$  is

$$\det(M) = \Lambda^{\text{rk}(M)}(M) \in \text{Proj}_1(R)$$

This defines a functor  $\det: \text{Proj}(R)^\simeq \rightarrow \text{Proj}_1(R)$ .

Remark This is a categorification of the determinant of matrices:

$$f: M \rightarrow M, \quad M \in \text{Proj}(R), \quad \det(f): \det(M) \rightarrow \det(M).$$

$$\det(f) \in \text{End}_R(\det(M)) \xrightarrow{\cong} R$$

$$f \text{ is invertible} \Leftrightarrow \det(f) \in R^\times.$$

Lemma:  $\det(P \oplus Q) \simeq \det(P) \otimes \det(Q)$ .

Pf:  $r = \text{rk}(P), s = \text{rk}(Q)$ . (assume these are constant).

$$\det(P \oplus Q) = \Lambda^{r+s}(P \oplus Q) \simeq \bigoplus_{\substack{i+j=r+s \\ i \leq r \text{ and } j \leq s}} \underbrace{\Lambda^i(P) \otimes \Lambda^j(Q)}_{=0 \text{ unless}} = \Lambda^r(P) \otimes \Lambda^s(Q) = \det(P) \otimes \det(Q). \quad \square$$

Warning:  $\det: (\text{Proj}(R)^\simeq, \oplus) \rightarrow (\text{Proj}_1(R), \otimes)$

$\pi$  is monoidal functor, but NOT symm. monoidal:

$$\det(P \oplus Q) \simeq \det(P) \otimes \det(Q).$$

$$\det(Q \oplus P) \simeq \det(Q) \otimes \det(P) \quad \begin{matrix} \text{can} & \text{(1 can} & \text{only commutes up} \\ \text{to a sign} & \text{to a sign} & \text{to a sign} \end{matrix} (-1)^{\text{rk}(P)\text{rk}(Q)}$$

Thus:

$$\begin{array}{ccc} \pi_0(\text{Proj}(R)^\simeq) & \xrightarrow{\det} & \pi_0(\text{Proj}_1(R)^\simeq) = \text{Pic}(R) \\ & \downarrow & \dashleftarrow \dashrightarrow \\ & K_0(R) & \dashleftarrow \exists! \det \end{array}$$

morphism  
of monoids

$\det$  is surjective: In fact  $(\text{rk}, \det): K_0(R) \xrightarrow{\sim} \text{Map}(\text{Spec } R, \mathbb{Z}) \times \text{Pic}(R)$

has a section:  $(n, L) \mapsto [R^n] + ([L] - [R]) \in K_0(R)$   
(not additive)

Def:  $S\mathcal{K}_0(R) = \text{kernel of } (\text{rk}, \det) \subset \tilde{\mathcal{K}}_0(R) \subset \mathcal{K}_0(R)$

Exercise:  $S\mathcal{K}_0(R) \subset \mathcal{K}_0(R)$  is an ideal.

Remark: The functors  $\text{Sym}^n, \Gamma^n, \Lambda^n : \text{Proj}(R) \rightarrow \text{Proj}(R)$  are not additive, but one can show that they induce maps  $\mathcal{K}_0(R) \rightarrow \mathcal{K}_0(R)$ .

The Hattori-Stallings trace (Dennis trace) ( $R$  arbitrary ring)

If  $M$  is a left  $R$ -module, then  $\text{Hom}_R(M, R)$  has a right  $R$ -module structure:  
 $(\varphi, r) \mapsto \varphi(-)r$

$$\begin{array}{ccc} \text{Hom}_R(M, R) \otimes_R M & \xrightarrow{\text{ev}} & R \\ \downarrow & & \downarrow \\ \text{Hom}_R(M, R) \otimes_R M & \xrightarrow{\text{ev}} & R/[R, R] \end{array}$$

$\text{ev}(\varphi \otimes x) = \varphi(x)$   
 $\text{ev}(\varphi \otimes rx) = \varphi(rx) = r\varphi(x)$   
 $\text{ev}(\varphi r \otimes x) = \varphi(x)r$

$[R, R] \subset R$  is the subgroup generated by  $rs - sr$ .

On the other hand, we have  $\text{Hom}_R(M, R) \otimes_R M \rightarrow \text{End}_R(M)$   
 $\varphi \otimes x \mapsto \varphi(-)x$ .

Lemma This is an isomorphism if  $M$  is  $f.g.$ -proj.

Pf: Clear if  $M = R^n$ . In general,  $M$  is a retract of  $R^n$ .  $\square$

Def The trace for  $P$  a  $f.g.$ -proj.  $R$ -module  $B$ :

$$\text{Tr} : \text{End}_R(P) \xleftarrow{\sim} \text{Hom}_R(P, R) \otimes_R P \xrightarrow{\text{ev}} R/[R, R].$$

Concretely: If  $P \oplus P' = R^n$ ,  $\text{Tr}(f: P \rightarrow P')$  is the trace of the matrix  $f \oplus 0_{P'}: R^n \rightarrow R^n$

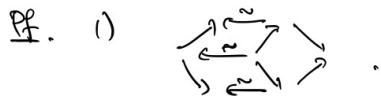
Properties of the trace

1) Naturality:  $u: P \rightarrow Q$

$$\begin{array}{ccc} \text{Hom}_R(Q, P) & \xrightarrow{u_*} & \text{End}_R(Q) \\ \uparrow & & \downarrow \text{Tr} \\ \text{End}_R(Q) & \xrightarrow{\text{Tr}} & R/[R, R] \end{array}$$

2) Additivity:  
 $\bullet f, g: P \rightarrow P \Rightarrow \text{Tr}(f+g) = \text{Tr}(f) + \text{Tr}(g)$   
 $\bullet f: P \rightarrow P, g: Q \rightarrow Q \Rightarrow \text{Tr}(f \oplus g) = \text{Tr}(f) + \text{Tr}(g)$

special cases  
of 1) { 3) Cyclicity:  $f, g: P \rightarrow P : \text{Tr}(f \circ g) = \text{Tr}(g \circ f)$   
4) Conjugation invariance:  $f: P \rightarrow P, g: P \xrightarrow{\sim} Q : \text{Tr}_P(f) = \text{Tr}_Q(g \circ f \circ g^{-1})$ .



□

Def: The Hectori-Shallings trace is

$$\begin{array}{ccc} \pi_0(\text{Proj}(R)^\cong) & \longrightarrow & R/[R,R] \\ \downarrow P & \xrightarrow{\quad} & \text{Tr}(\text{id}_P) \\ K_0(R) & \xrightarrow{\exists! \text{ Tr}} & \end{array} \quad \begin{array}{l} \text{well-defined morphism} \\ \text{of comm. monoids} \end{array}$$

### Remarks

- 1) If  $R$  commutative, this is almost the rank map:

$$\begin{array}{ccc} K_0(R) & \xrightarrow{\text{Tr}} & R \\ \text{rk} \downarrow & \nearrow & \text{the unique map } \mathbb{Z} \rightarrow R, \text{ extended to} \\ \text{Maps}(\text{Spec } R, \mathbb{Z}) & & \text{locally constant integers.} \end{array}$$

$$\begin{array}{c} \text{Def: If } p \subset R \text{ prime ideal, } R_p \text{ local } \Rightarrow K_0(R_p) \xrightarrow{\text{rk}} \text{Maps}(\text{Spec } R_p, \mathbb{Z}) \cong \mathbb{Z} \\ \uparrow \quad \uparrow \text{ev}_p \\ K_0(R) \xrightarrow{\text{rk}} \text{Maps}(\text{Spec } R, \mathbb{Z}) \\ \text{ } \\ 0 \rightarrow \tilde{K}_0(R) \rightarrow K_0(R) \rightarrow \prod_{p \subset R} K_0(R_p) \\ \text{Tr} \downarrow \quad \downarrow \text{Tr} \quad \Rightarrow \text{Tr}|_{\tilde{K}_0(R)} = 0. \quad \square \\ R \hookrightarrow \prod_p R_p \end{array}$$

- 2)  $R/[R,R]$  is also known as  $\text{HH}_0(R)$ , the zeroth Hochschild homology of  $R$ . The Dennis trace  $K_*(R) \rightarrow \text{HH}_*(R)$  is a generalization of the H-S trace to higher K-theory. It is one of the main sources of computations in K-theory.