

### Stably free modules

Def. An  $R$ -module  $M$  is stably free (of rank  $n-m$ ) if  $M \oplus R^m \cong R^n$ .

( $R$  comm.) Recall:  $K_0(R) \cong \text{Map}_0(\text{Spec } R, \mathbb{Z}) \oplus \tilde{K}_0(R)$   
and  $\tilde{K}_0(R) = \varinjlim_{n \rightarrow \infty} \pi_0(\text{Proj}_n(R))$ .

$M$  stably free  $\iff M$  has constant rank and becomes 0 in  $\tilde{K}_0(R)$   
 $\iff [M] = [R^n]$  for some  $n$  in  $K_0(R)$ .

$\Rightarrow$  K-theory detects stably free modules.

Question: When does stably free  $\Rightarrow$  free? stably isomorphic  $\Rightarrow$  isomorphic?

Example: stably free  $\not\Rightarrow$  free in general (cf. Exercises).

Example: Stably isomorphic line bundles are isomorphic:

$$L \oplus R^n \cong L' \oplus R^n \Rightarrow L \cong \det(L \oplus R^n) \cong \det(L' \oplus R^n) \cong L'.$$

Def.  $X$  top. space. The Krull dimension of  $X$  is

$$\dim(X) = \sup \{ n \mid \exists \text{ chain } Z_0 \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n \subset X \text{ of irreducible closed subsets of } X \}$$

If  $R$  comm. ring,  $\dim(R) = \dim(\text{Spec } R)$ .

Thm (Bass-Serre cancellation theorem)

Let  $R$  be a noetherian comm. ring of Krull dimension  $d$ , and  $P$  a fg. proj.  $R$ -module of rank  $> d$ .

1) (Serre)  $P \cong P' \oplus R$  for some  $P'$  (hence:  $P \cong P' \oplus R^{\text{rk}(P)-d}$ )

2) (Bass) If  $P$  is stably isomorphic to  $Q$ , then  $P \cong Q$ .

Corollary Let  $R$  be a noeth. comm. ring of  $\dim \leq 1$ , e.g. a Dedekind domain.

If  $P \in \text{Proj}(R)$ ,  $P \neq 0$ , then  $P \cong \det(P) \oplus R^{\text{rk}(P)-1}$ .

PF. We have  $P \cong L \oplus \text{free}$  by Serre, so  $L = \det(L \oplus \text{free}) = \det(P)$ .  $\square$

Corollary If  $R$  is noeth. of  $\dim 1$ , then  $(\text{rk}, \det): K_0(R) \xrightarrow{\cong} \text{Map}(\text{Spec } R, \mathbb{Z}) \times \text{Pic}(R)$   
In other words,  $\mathcal{SK}_0(R) = 0$ .

K<sub>1</sub>

- Def R ring.
- $GL(R) = \varinjlim_n GL_n(R)$   $GL_n(R) \rightarrow GL_{n+1}(R)$
  - $E_n(R) \subset GL_n(R)$  subgroup generated by elementary matrices  $e_{ij}(r) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & r & \\ & & & 1 \end{pmatrix}_{i \neq j}$
  - $E(R) = \varinjlim_n E_n(R) \subset GL(R)$
  - $K_1(R) = GL(R)/E(R) = GL(R)^{ab}$
- Exercise:  $E(R) = [GL(R), GL(R)]$ .

Q: Why?

Idea:

$$\begin{array}{ccc} Proj(R)^\simeq & \xrightarrow{\pi_0} & \pi_0(Proj(R)^\simeq) \\ \downarrow & & \downarrow \text{group completion} \\ ? & \xrightarrow{\pi_0} & K_0(R). \end{array}$$

Repeat the construction of  $K_0$  without taking  $\pi_0$  first.  
i.e., do "group completion" at the groupoid level.

A more basic example: Fin = category of finite sets

$$\begin{array}{ccc} Fin^\simeq & \longrightarrow & \pi_0(Fin^\simeq) \cong \mathbb{N} \\ \downarrow & & \downarrow \\ ? & \longrightarrow & \mathbb{Z} \end{array}$$

Groupoids (= category where all morphisms are invertible)

- Example:
- Every set can be viewed as a groupoid (with only identity morphisms)
  - If  $G$  is a group, we can define a groupoid  $BG$  with one object  $*$  and  $\text{Aut}_\text{BG}(*) = G$ .

Recall: a functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence if it is fully faithful and essentially surjective.

Def.  $\mathcal{C}$  groupoid.

- $\pi_0(\mathcal{C}) =$  set of isomorphism classes of objects of  $\mathcal{C}$ .
- for  $X \in \mathcal{C}$ ,  $\pi_1(\mathcal{C}, X) = \text{Aut}_{\mathcal{C}}(X)$  (group).
- if  $\mathcal{C}, \mathcal{D}$  are groupoids, we can form a groupoid  $\text{Map}(\mathcal{C}, \mathcal{D}) = \text{Fun}(\mathcal{C}, \mathcal{D})$ :
  - objects are functors
  - morphisms are natural transformations

Observation: every groupoid is equivalent to one of the form  $\coprod_{i \in I} BG_i$  for some set  $I$  and some groups  $G_i$ .

Pf. Let  $\mathcal{C}$  be a groupoid. Choose a set of representatives  $\{x_i\}_{i \in I}$  of iso classes in  $\mathcal{C}$ . Let  $G_i = \text{Aut}_{\mathcal{C}}(x_i)$ . Then we get

$$\coprod_{i \in I} BG_i \longrightarrow \mathcal{C}$$

$$*_i \longmapsto x_i$$

By construction, this is fully faithful and essentially surjective.

Observation if  $f: \mathcal{C} \rightarrow \mathcal{D}$  is a equivalence of groupoids then

- $\pi_0(f): \pi_0(\mathcal{C}) \xrightarrow{\cong} \pi_0(\mathcal{D})$
- $\forall x \in \mathcal{C}, \pi_1(f, x): \pi_1(\mathcal{C}, x) \xrightarrow{\cong} \pi_1(\mathcal{D}, f(x))$ .
- if  $\Sigma$  is any groupoid,  $\text{Map}(\Sigma, \mathcal{C}) \xrightarrow{f_*} \text{Map}(\Sigma, \mathcal{D})$  is an equivalence.  
 $\text{Map}(\mathcal{D}, \Sigma) \xrightarrow{f^*} \text{Map}(\mathcal{C}, \Sigma)$ , is an equivalence.

Theorem Top category of homological spaces.

A map  $f: X \rightarrow Y$  in Top is a weak equivalence if it induces isomorphisms on all homotopy groups.

A top space  $X \in \text{Top}$  is  $n$ -truncated if  $\pi_i(X, x) = 0$  for  $i > n$  and  $x \in X$ :

$\text{Top}_{\leq n} \subset \text{Top}$  full subcategory of  $n$ -truncated spaces.

$\text{Top}_{\leq 0}[\text{w.e.}] \xrightarrow{\pi_0} \text{Set}$  is an equivalence.

$\text{Top}_{\leq 1}[\text{w.e.}] \xrightarrow{\pi_1} \text{Gpd}[\text{equi}^{-1}]$ .

$\pi_1 X$  - objects are points  
 $\times_{X \in X}$   
 morphisms are  
 homotopy classes  
 of paths.

Philosophical points We regard two groupoids as "the same" if we are given an equivalence between them.

e.g.  and  are equivalent.

$f \circ g = \text{id}$   $g \circ f = \text{id}$

so "# of objects" is not a well-defined property of a groupoid.