

Symmetric monoidal categories

Def.

- 1) A monoidal category is a category \mathcal{C} with a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and an object $\mathbb{1} \in \mathcal{C}$, and natural isomorphisms:

$$\alpha_{X,Y,Z}: (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

$$\lambda_X: \mathbb{1} \otimes X \xrightarrow{\cong} X$$

$$\rho_X: X \otimes \mathbb{1} \xrightarrow{\cong} X$$

satisfying the following axioms:

- Unit axiom: $(X \otimes \mathbb{1}) \otimes Y \xrightarrow{\alpha} X \otimes (\mathbb{1} \otimes Y)$

$$\rho_X^{-1} \quad \text{---} \quad \lambda_Y$$

- Pentagon axiom: $((X \otimes Y) \otimes Z) \otimes W \xrightarrow{\alpha} (X \otimes Y) \otimes (Z \otimes W)$

$$\text{---} \quad \text{---}$$

$$(X \otimes (Y \otimes Z)) \otimes W \quad \text{---} \quad X \otimes (Y \otimes (Z \otimes W))$$

$$\lambda_Z$$

$$\text{---} \quad \text{---}$$

$$X \otimes ((Y \otimes Z) \otimes W)$$

- 2) A braided monoidal category is a monoidal category with an additional natural isomorphism $\gamma_{X,Y}: X \otimes Y \xrightarrow{\cong} Y \otimes X$ such that:

- $X \otimes \mathbb{1} \xrightarrow{\gamma} \mathbb{1} \otimes X$

$$\rho_X^{-1} \quad \text{---} \quad \lambda_X$$

- Hexagon axiom: $(X \otimes Y) \otimes Z \xrightarrow{\alpha} X \otimes (Y \otimes Z)$

$$\text{---} \quad \text{---}$$

$$(Y \otimes X) \otimes Z \quad \text{---} \quad (Y \otimes Z) \otimes X$$

$$\lambda_Z$$

$$\text{---}$$

$$Y \otimes (X \otimes Z) \xrightarrow{\gamma} Y \otimes (Z \otimes X)$$

- 3) A symmetric monoidal category is a braided monoidal category where $\gamma^2 = \text{id}$.

- 4) If \mathcal{C} & \mathcal{D} are monoidal categories, a monoidal functor from \mathcal{C} to \mathcal{D}
 is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ with natural isomorphisms:

$$F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{\epsilon} \mathbb{1}_{\mathcal{D}} \quad F(X \otimes Y) \xrightarrow{\mu} F(X) \otimes F(Y)$$

and that:

- $F((X \otimes Y) \otimes Z) \xrightarrow{\text{Nat } F} F(X \otimes (Y \otimes Z))$
 $F(X \otimes Y) \otimes F(Z) \xrightarrow{\text{Nat } F} F(X) \otimes F(Y \otimes Z)$
 $(F(X) \otimes F(Y)) \otimes F(Z) \xrightarrow{\text{Nat } F} F(X \otimes (F(Y) \otimes F(Z)))$
- $\mathbb{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon} F(\mathbb{1}_{\mathcal{C}}) \otimes F(X)$
 $\lambda \text{ Nat } \circ \text{ Nat } \mu \quad \text{and similarly for } \rho.$
 $F(X) \xrightarrow{\text{Nat } F} F(\mathbb{1}_{\mathcal{C}} \otimes X)$

- 5) A braided monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor
 such that

$$\begin{aligned} F(X \otimes Y) &\xrightarrow{F(\eta)} F(Y \otimes X) \\ \mu \text{ Nat } &\circ \text{ Nat } \mu \\ F(X) \otimes F(Y) &\xrightarrow{\text{Nat } F} F(Y) \otimes F(X) \end{aligned}$$

- 6) A sym.-monoidal functor is a braided monoidal functor.

- 7) If $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are monoidal functors, a monoidal natural transformation $\varphi: F \rightarrow G$ is a natural transformation s.t.:

$$\begin{array}{ccc} F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{\varphi} G(\mathbb{1}_{\mathcal{C}}) & F(X \otimes Y) \xrightarrow{\varphi} G(X \otimes Y) \\ \epsilon \text{ Nat } \circ \text{ Nat } \varphi & \mu \text{ Nat } \circ \text{ Nat } \mu \\ \mathbb{1}_{\mathcal{D}} & F(X) \otimes F(Y) \xrightarrow{\varphi \otimes \varphi} G(X) \otimes G(Y) \end{array}$$

- 8) & 9) A braided/sym.-monoidal natural transformation \Rightarrow
 • monoidal natural transformations.

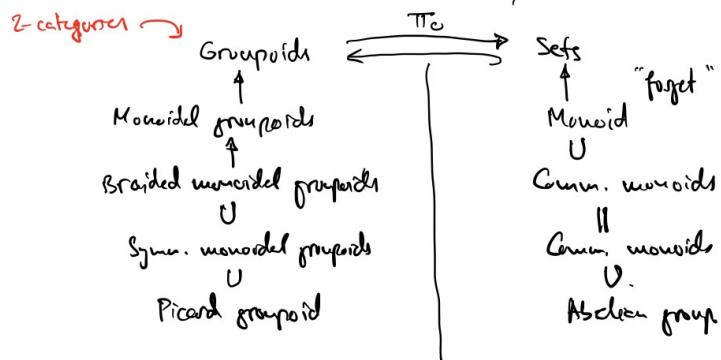
Notation If \mathcal{C}, \mathcal{D} are sym. monoidal categories, $\text{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ is the category of sym. monoidal functors and natural transformations.

Def. A Picard groupoid is a symmetric monoidal groupoid in which every object has a \otimes -inverse: $\forall X, \exists X' \text{ s.t. } X \otimes X' = \mathbb{1}$.

Example If \mathcal{C} is a sym., monoidal category, $\underline{\text{Pic}}(\mathcal{C}) \subset \mathcal{C}^{\cong}$ is the full subcategory of \otimes -invertible objects. Then $\underline{\text{Pic}}(\mathcal{C})$ is a Picard groupoid.

e.g. R comm. ring, $\underline{\text{Pic}}(\text{Mod}_R, \otimes) = \text{Proj}(R)^{\cong}$.
 π_0 is $\text{Pic}(R)$. $\pi_1, \underline{\text{Pic}}(\text{Mod}_R) \simeq R^X$

Remark: If \mathcal{C} is a Picard groupoid, $\pi_1(\mathcal{C}, X) \xleftarrow[\sim]{(-\otimes X)} \pi_1(\mathcal{C}, \mathbb{1})$
 $(-\otimes X: \mathcal{C} \rightarrow \mathcal{C} \text{ is an equivalence})$



Def. let \mathcal{C} be a sym. monoidal groupoid. A group completion of \mathcal{C} is a Picard groupoid \mathcal{C}^{op} with a sym. monoidal functor $\eta: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ satisfying the following universal property:

$$\forall \text{ Picard groupoid } \mathcal{E}, \quad \text{Fun}^{\otimes}(\mathcal{C}^{\text{op}}, \mathcal{E}) \xrightarrow{\eta^*} \text{Fun}^{\otimes}(\mathcal{C}, \mathcal{E})$$

is an equivalence of categories.

If \mathcal{C}^{op} exists, then it is unique up to equivalence.

- Rmk:
- 1) 2-categorical adjoint functor theorem $\Rightarrow \mathcal{C}^{\text{op}}$ always exists.
 - 2) $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ also has a universal property as a monoidal functor.

Lemma. \mathcal{C} sym. monoidal groupoid. Then $\pi_0(\mathcal{C}^{\text{op}}) \cong \pi_0(\mathcal{C})^{\text{op}}$.

Pf. Compare universal properties: for any abelian group A ,

$$\text{Hom}_{\text{Ab}}(\pi_0(\mathcal{C}^{\text{op}}), A) = \text{Fun}^{\otimes}(\mathcal{C}^{\text{op}}, A) \xrightarrow[\text{discrete.}]{} \text{Fun}^{\otimes}(\mathcal{C}, A) = \text{Hom}_{\text{CMon}}(\pi_0(\mathcal{C}), A)$$

$$\cong \text{Hom}_{\text{Ab}}(\pi_0(\mathcal{C})^{\text{op}}, A).$$

OR: $\text{Ab} \hookrightarrow \text{CMon}$ 2-categorical

$$\begin{matrix} \downarrow & \downarrow \\ \text{PicGpd} & \hookrightarrow \text{SymMonGpd.} \end{matrix} \Rightarrow \text{the squares of left-adjoints commute.}$$

Def. Let R be a ring. The K -theory groupoid of R is

$$\tau_{\leq 1} K(R) = (\text{Proj}(R)^{\cong}, \oplus)^{\text{op}}$$

By the lemma, $\pi_0(\tau_{\leq 1} K(R)) \cong K_0(R)$

Def. $K_1(R) = \pi_1(\tau_{\leq 1} K(R), 0)$

Proposition (the Eckmann-Hilton argument)

Let \mathcal{C} be a monoidal groupoid. Then the group $\pi_1(\mathcal{C}, \mathbb{1})$ is abelian.

Proof. $\pi_1(\mathcal{C}, \mathbb{1}) = \text{Aut}_{\mathcal{C}}(\mathbb{1})$ has two multiplications:

composition: $\circ : \pi_1(\mathcal{C}, \mathbb{1}) \times \pi_1(\mathcal{C}, \mathbb{1}) \rightarrow \pi_1(\mathcal{C}, \mathbb{1})$

tensor product: $\otimes : \pi_1(\mathcal{C}, \mathbb{1}) \times \pi_1(\mathcal{C}, \mathbb{1}) \rightarrow \pi_1(\mathcal{C}, \mathbb{1})$

$$(f: \mathbb{1} \rightarrow \mathbb{1}, g: \mathbb{1} \rightarrow \mathbb{1}) \mapsto \begin{array}{c} \mathbb{1} \\ \downarrow \lambda = f \\ \mathbb{1} \otimes \mathbb{1} \end{array} \xrightarrow{\quad \text{fog} \quad} \begin{array}{c} \mathbb{1} \\ \downarrow \lambda = g \\ \mathbb{1} \otimes \mathbb{1} \end{array}$$

Since $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a functor, we have:

$$(*) \quad (f \circ g) \otimes (h \circ k) = (f \otimes h) \circ (g \otimes k) \quad \forall f, g, h, k \in \pi_1(\mathcal{C}, \mathbb{1})$$

Also: $f \otimes \text{id}_{\mathbb{1}} = f$ and $\text{id}_{\mathbb{1}} \otimes f = f$. in $\pi_1(\mathcal{C}, \mathbb{1})$

$$\begin{array}{ccc} & \nearrow & \\ \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\text{f} \otimes \text{id}} & \mathbb{1} \otimes \mathbb{1} \\ \downarrow \rho & & \downarrow \rho \\ \mathbb{1} & \xrightarrow{f} & \mathbb{1} \end{array} \quad \text{commutes by naturality of } f.$$

Now:

$$f \circ g = (\text{id} \otimes f) \circ (g \otimes \text{id}) \stackrel{(*)}{=} (\text{id} \circ g) \otimes (f \circ \text{id}) = g \circ f$$

$$g \circ f = (g \otimes \text{id}) \circ (\text{id} \otimes f) \stackrel{(*)}{=} (g \circ \text{id}) \otimes (\text{id} \circ f) = g \circ f \quad \square$$

Exercise: \exists equivalences of categories:

- $\text{CMon}(\text{Set}) \simeq \text{Mon}(\text{Mon}(\text{Set}))$
- $\text{Mon}(\text{CMon}(\text{Set})) \simeq \text{CMon}(\text{Set}).$

Remark. In 2-category theory, one can define monoids, and we have:

- $\text{Mon}(\text{Cat}) \simeq \text{MonCat}$
- $\text{Mon}(\text{Mon}(\text{Cat})) \simeq \text{BrMonCat}$
- $\text{Mon}(\text{Mon}(\text{Mon}(\text{Cat}))) \simeq \text{SymMonCat}.$
- $\text{Mon}^{(n)}(\text{Cat}) \simeq \text{SymMonCat}$ for all $n \geq 3$.

Example : free monoidal categories

- $(\mathbb{N}, +)$ as a monoidal category is the free monoidal category on one object:
A monoidal category \mathcal{C} ,

$$\text{Fun}^{\text{mon}}(\mathbb{N}, \mathcal{C}) \xrightarrow{\text{ev}_1} \mathcal{C} \text{ is an equivalence.}$$

- $(\text{Fin}^{\cong}, \amalg, \emptyset)$ is the free symmetric monoidal category on one object:

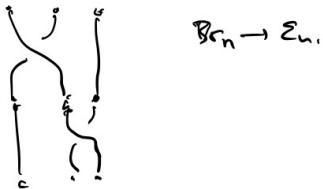
A sym. monoidal cat. \mathcal{C} :

$$\text{Fun}^{\text{sym.mon}}(\text{Fin}^{\cong}, \mathcal{C}) \xrightarrow{\text{ev}_{\text{fr}}} \mathcal{C} \text{ is an equivalence.}$$

Note: $\text{Fin}^{\cong} = \coprod_{n \in \mathbb{N}} B\Sigma_n$

- the free braided monoidal category on one object is $\coprod_{n \in \mathbb{N}} BBr_n$
where Br_n is the braid group on n strands.

$n=3$



$$Br_3 \rightarrow \Sigma_3.$$