

A toy example: finite sets      Let  $\mathcal{F} = \text{Fin}^{\approx}$ .

Goal: compute the Picard groupoid  $\mathcal{F}^{\text{op}}$ .

$$\text{Recall: } \mathcal{F} \simeq \coprod_{n \in \mathbb{N}} B\Sigma_n$$

$$\text{We know: 1) } \pi_0(\mathcal{F}^{\text{op}}) \simeq \pi_0(\mathcal{F})^{\text{op}} = \mathbb{N}^{\text{op}} \simeq \mathbb{Z}.$$

$$2) \quad \pi_1(\mathcal{F}^{\text{op}}, x) \simeq \pi_1(\mathcal{F}^{\text{op}}, \emptyset) \text{ is abelian.}$$

$$\Rightarrow \mathcal{F}^{\text{op}} \simeq \coprod_{n \in \mathbb{Z}} BA \quad \text{for some abelian group } A = \pi_1(\mathcal{F}^{\text{op}}, \emptyset)$$

What is A?

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\eta} & \mathcal{F}^{\text{op}} \\ \Downarrow \text{sign} \downarrow & \hookrightarrow & \downarrow \text{sign} \\ \mathcal{F} & \xrightarrow{\eta} & \mathcal{F}^{\text{op}} \end{array}$$

$$\begin{array}{ccc} & & \uparrow \{1, \dots, n\} \\ & & \Sigma_n = \pi_1(\mathcal{F}, n) \longrightarrow \pi_1(\mathcal{F}^{\text{op}}, n) \simeq \\ \nearrow \pi_1 \downarrow & & \downarrow \hookrightarrow \quad \downarrow \simeq \quad \downarrow \simeq \\ \Sigma_{n+1} = \pi_1(\mathcal{F}, n+1) & \longrightarrow & \pi_1(\mathcal{F}^{\text{op}}, n+1) \simeq \end{array}$$

$$\text{and} \quad \underset{n \rightarrow \infty}{\text{colim}} \Sigma_n \longrightarrow A \\ \Sigma_{\infty} \longrightarrow \Sigma_{\infty}^{\text{ab.}}$$

Fact: If  $n \geq 2$ ,  $\Sigma_n^{\text{ab.}} \simeq C_2$  via the sign homomorphism

$$(\Sigma_n, \Sigma_n) = A_n$$

$$\text{sign: } \Sigma_n \longrightarrow \{ \pm 1 \} = C_2$$

$$\Rightarrow \Sigma_{\infty}^{\text{ab.}} \simeq C_2 \text{ via sign: } \Sigma_{\infty} \longrightarrow C_2.$$

This suggests  $A = C_2$ . How can we prove this?

We have a factorization of  $\eta: \mathcal{F} \rightarrow \mathcal{F}^{\text{op}}$  as

$$\coprod_{n \in \mathbb{N}} B\Sigma_n \xrightarrow{\text{sign}} \coprod_{n \in \mathbb{Z}} BC_2 \longrightarrow \coprod_{n \in \mathbb{Z}} BA$$

$\rightarrow$  Promote this to a symmetric monoidal factorization.

Construction the Picard groupoid  $\mathbb{S}$ .

- objects  $n \in \mathbb{Z}$ .
- morphism:  $\text{Hom}_{\mathbb{S}}(n, m) = \emptyset$  if  $n \neq m$  and  $\text{Aut}_{\mathbb{S}}(n) = \{\pm 1\}$
- unit object:  $0 \in \mathbb{Z}$ .
- $\otimes: \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$   $(n, m) \mapsto n+m$
- $\text{Aut}(n) \times \text{Aut}(m) \rightarrow \text{Aut}(n+m)$   
 $\{\pm 1\} \times \{\pm 1\} \xrightarrow{\cdot} \{\pm 1\}$

- $d, p, \lambda$  are the identity.
- braiding:  $\gamma_{n,m}: n+m \cong n+m$
- $\gamma_{n,m} \in \text{Aut}(n+m) = \{\pm 1\}$   $\gamma_{n,m} = (-1)^{nm}$

Axioms:

$$\begin{array}{ccc} n+0 & \xrightarrow{\gamma_{n,0}} & 0+n \\ \parallel & & \parallel \\ n & & n \end{array} \quad \gamma_{n,0} = (-1)^{n0} = 1 \quad \checkmark$$

$$\begin{array}{ccc} (n+m)+k & \xrightarrow{\gamma_{n,m}} & (m+n)+k \\ \downarrow = & & \downarrow = \\ n+(m+k) & & m+(n+k) \\ \downarrow \gamma_{n,m+k} & & \downarrow \gamma_{n,k} \\ (m+k)+n & \xrightarrow{=} & m+(k+n) \end{array}$$

$$\begin{aligned} \gamma_{n,m} \cdot \gamma_{n,k} &= \gamma_{n,m+k} \\ (-1)^{nm}(-1)^{nk} &= (-1)^{n(m+k)}. \end{aligned} \quad \checkmark$$

$$\gamma^2 = \text{id} \quad \checkmark.$$

$\Rightarrow \mathbb{S}$  is a Picard groupoid.

Rmk. As a groupoid,  $\mathbb{S} \cong \mathbb{Z} \times \text{BC}_2$ .

Both  $\mathbb{Z}$  and  $\text{BC}_2$  are also Picard groupoids.

This equivalence is monoidal but not symmetric monoidal.

Claim:  $\coprod_{n \in \mathbb{N}} \text{BS}_n \xrightarrow{\text{sign}} \mathbb{S}$  is a symmetric monoidal functor  
(with  $\varepsilon, \mu$  the identity)

$\coprod_{n \in \mathbb{N}} \text{BS}_n \xrightarrow{S} \mathbb{S}$  What is the symmetric monoidal structure on  $\coprod_{n \in \mathbb{N}} \text{BS}_n$ ?

- $\alpha, \lambda, \rho$  are the identity
- $\gamma_{n,m} \in \text{Aut}(n+m) = \Sigma_{n+m}$   $\gamma_{n,m} = ?$

$$s(n+m) \xrightarrow{\mu} s(n) \amalg s(m).$$

$$\{1, \dots, n+m\} \xrightarrow{\quad i \quad} \{1, \dots, n\} \amalg \{1, \dots, m\} \xrightarrow{\quad \phi \quad} \{1, \dots, m\}$$

$i$

$j+n \longleftarrow \quad \quad \quad j$

$$\{1, \dots, n\} \amalg \{1, \dots, m\} \xrightarrow{\text{swap}} \{1, \dots, m\} \amalg \{1, \dots, n\}.$$

$\cong 12$

$$\{1, \dots, n+m\} \xrightarrow{\quad \theta_{n,m} \quad} \{1, \dots, n+m\} \xrightarrow{\quad 12 \text{ P.} \quad}$$

$\Rightarrow \theta_{n,m}$  is the  $(n,m)$ -shuffle:

$$\text{sign}(\theta_{n,m}) = (-1)^{nm} = \theta_{n,m} \text{ in } \mathbb{S}.$$

$$\Rightarrow \text{sign}: \coprod_{n \in \mathbb{N}} B\Sigma_n \longrightarrow \mathbb{S} = \coprod_{n \in \mathbb{Z}} BC_n \text{ is symm. monoidal.}$$

Also:  $\mathbb{S} \longrightarrow \coprod_{n \in \mathbb{Z}} BA$  is symm monoidal (tedious verification).

$$\Rightarrow \begin{array}{ccc} \coprod_{n \in \mathbb{N}} B\Sigma_n & \xrightarrow{\exists \eta} & \mathbb{S} \\ \text{id}_{\mathbb{S}} \downarrow & \nearrow \exists ! \simeq & \downarrow \text{id.} \\ & \text{id}_{\mathbb{S}} & \end{array} = C_2 \overset{\sim}{\hookrightarrow} A$$

Conclusion:  $\mathbb{F}^{\text{op}} \simeq \mathbb{S}$  as Picard groupoids.

This computation generalizes as follows.

Def: Let  $M$  be a comm. monoid,  $L \subset M$  submonoid. We say that  $L$  is cofinal in  $M$  if  $\forall x \in M, \exists y \in M$  s.t.  $xy \in L$ .

Theorem (Special case of the "group completion theorem")

Let  $(\mathcal{C}, \oplus)$  be a symm. monoidal groupoid. Suppose there is a cofinal embedding  $\mathbb{N} \subset \pi_0(\mathcal{C})$ . Let  $G_n = \text{Aut}_{\mathcal{C}}(n)$  for  $n \in \mathbb{N}$ , and let  $G_{\infty} = \varprojlim G_n$  where

$$G_n \hookrightarrow G_n \times G_1 \xrightarrow{\oplus} G_{n+1}$$

$$f \mapsto (f, \text{id})$$

Then there is an equivalence of groupoids  $\mathcal{C}^{\text{op}} \simeq \coprod_{\pi_0(\mathcal{C})^{\text{op}}} BG_{\infty}^{\text{ab}}$ .

Pf. Essentially the same as for  $C = \text{Fin}^{\infty}$ .

Remarks. 1) The equivalence  $C^{\text{op}} \simeq \pi_0(C)^{\text{op}} \times BG_{\infty}^{\text{ab}}$  is monoidal but not symmetrisable in general.

2) If  $N \subset M$  is cofinal, then

$$M^{\text{op}} \simeq \text{colim}(M \xrightarrow{+1} M \xrightarrow{+1} M \rightarrow \dots)$$

(Exercise).

For  $C$  as in the theorem,

$$\text{colim}(C \xrightarrow{+1} C \xrightarrow{+1} C \rightarrow \dots) = \coprod_{\pi_0(C)^{\text{op}}} BG_{\infty}$$

This does not admit a monoidal structure in general, since  $\pi_1 = G_{\infty}$  may not be abelian. (by Eckmann-Hilton)

3) The assumption on  $\pi_0(C)$  can be removed:

$\pi_0(C) = M$ . Choose a set of generators  $\{m_i \mid i \in I\} \subset M$   
 and  $\varphi: N^{(I)} \rightarrow M$

$$G_{\infty} = \underset{\vec{n} \in N^{(I)}}{\text{colim}} \text{Aut}_C(\varphi(\vec{n})) \quad \text{filtered colimit.}$$

$$\Rightarrow C^{\text{op}} \simeq \coprod_{\pi_0(C)^{\text{op}}} BG_{\infty}^{\text{ab}}.$$

Cor Let  $R$  be a ring,  $\tau_{\leq 1} K(R) = (\text{Proj}(R)^{\infty}, \oplus)^{\text{op}} \simeq \coprod_{K_0(R)} BGL(R)^{\text{ab}}$

where  $GL(R) = \text{colim}_n GL_n(R)$

In particular,  $K_1(R) = GL(R)^{\text{ab}} = \bigcap_{\mathbb{Z}} GL(R)/E(R)$   
 cf. exercise.