

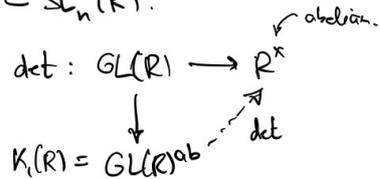
If  $R$  is a commutative, and  $P \in \text{Proj}(R)$ , we have

$$\det : \text{Aut}_R(P) \rightarrow R^\times$$

Def.  $SL_n(R) = \ker(\det : GL_n(R) \rightarrow R^\times)$

$$\det(e_j(r)) = 1 \Rightarrow E_n(R) \subset SL_n(R).$$

$$\det \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \det(A) \Rightarrow \det : GL(R) \rightarrow R^\times$$



Def.  $SK_1(R) = \ker(\det : K_1(R) \rightarrow R^\times)$     i.e.  $SK_1(R) = SL(R)/E(R).$

$$0 \rightarrow SK_1(R) \rightarrow K_1(R) \xrightarrow{\det} R^\times \rightarrow 0$$

$$\begin{matrix} \leftarrow \uparrow \\ \left( \begin{smallmatrix} r & 0 \\ 0 & 1 \end{smallmatrix} \right) \leftarrow r \end{matrix}$$

$$\Rightarrow K_1(R) = SK_1(R) \oplus R^\times.$$

Remark. (The symm. unimodular determinant)

Recall that  $\det$  is a non-symm. unimodular functor  $\text{Proj}(R) \cong \rightarrow \text{Pic}(\text{Mod}_R) \cong \text{Pic}(R)$ .  
We can make it symm. unimodular as follows:

let  $\text{Pic}^{\mathbb{Z}}(R)$  be the groupoid with:

object: pairs  $(L, n)$  with  $L \in \text{Pic}(R)$  and  $n : \text{Spec } R \rightarrow \mathbb{Z}$  continuous map.

morphism:  $\text{Hom}((L, n), (L', n')) = \begin{cases} \emptyset & \text{if } n \neq n' \\ \text{Isom}_R(L, L') & \text{if } n = n' \end{cases}$

$$\text{Thus } \text{Pic}^{\mathbb{Z}}(R) = \text{Pic}(R) \times \text{Maps}(\text{Spec } R, \mathbb{Z}).$$

symm unimodular structure:  $(L, n) \otimes (L', n') = (L \otimes L', n+n')$

braidry:  $\gamma_{(L, n), (L', n')} : (L \otimes L', n+n') \cong (L' \otimes L, n'+n)$   
is  $(-1)^{mn'} \gamma_{L, L'} : L \otimes L' \cong L' \otimes L$

One can verify that the unimodular functor

$$(\det, rk) : \text{Proj}(R) \cong \rightarrow \text{Pic}^{\mathbb{Z}}(R)$$

is symmetric. Since  $\text{Pic}^{\mathbb{Z}}(R)$  is a Picard groupoid, we get:

$$(\det, rk) : \pi_{\leq 1} K(R) \rightarrow \text{Pic}^{\mathbb{Z}}(R).$$

which recovers  $(\det, rk)$  on  $\pi_0$  and  $\det$  on  $\pi_1$ .

Rem:  $\text{Pic}^Z(R) = \text{Pic}(D(R))$

$K_i$  computations.

- if  $F$  is a field, we have  $E_n(F) = \text{SL}_n(F)$  by Gaussian elimination.  
Hence  $K_1(F) = \text{GL}(F)/\text{SL}(F) \xrightarrow[\det]{\cong} F^\times$ , equivalently  $\text{SK}_1(F) = 0$ .
- More generally, if  $D$  is a division ring,  $K_1(D) = D^\times / [D^\times, D^\times]$ .

Proposition If  $R$  is a comm. local ring,  $E_n(R) = \text{SL}_n(R)$  for  $n \geq 1$   
Hence  $\text{SK}_1(R) = 0$ ,  $K_1(R) = R^\times$ .

Pf. We know  $\begin{pmatrix} 1 & & \\ & \ddots & * \\ 0 & & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & \lambda & \ddots \\ & & \ddots & 1 \end{pmatrix} \in E_n(R)$   
(cf. exercise).

so it suffices to show that any  $g \in \text{GL}_n(R)$  can be put in upper triangular form using elementary row operations.

Key point: every row/column contains an invertible element.  
(obvious if  $R$  is a field, hence if  $R$  is local).

$$g = \begin{pmatrix} * \\ \vdots \\ \lambda \\ \vdots \\ * \end{pmatrix} \xrightarrow{e_{ii}(1) e_{ii}(-1) e_{ii}(1)} \begin{pmatrix} u & & \\ \vdots & * & \\ & & \vdots \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} u & & \\ 0 & \lambda & * \\ \vdots & & \\ 0 & & \end{pmatrix}$$

$\uparrow$   
~ upper triangular by induction.

Def. A comm ring  $R$  is a Euclidean domain if it is an integral domain and  $\exists$  function  $\text{deg}: R \setminus \{0\} \rightarrow \mathbb{N}$   
s.t.  $\forall a, b \in R, b \neq 0, \exists q, r \in R$  s.t.  
 $a = bq + r$  and either  $r = 0$  or  $\text{deg}(r) < \text{deg}(b)$ .

Rem Euclidean domain  $\Rightarrow$  PID. ( $\Leftarrow$ )

- Examples:
- fields
  - $\mathbb{Z}$  with  $\text{deg}(n) = |n|$
  - $k[x]$ ,  $k$  field,  $\text{deg} = \text{deg}$ .
  - $\mathbb{Z}[i]$  with  $\text{deg}(a+bi) = a^2 + b^2$ .
  - $\mathbb{Q}[\sqrt{d}]$   $d$  square-free, integer is a PID  $\Leftrightarrow$   
 $d = \underbrace{-1, -2, -3, -7, -11, -19, -43, -67, -163}_{\text{Euclidean}}, \underbrace{\dots}_{\text{not Euclidean}}$

Thm If  $R$  is a Euclidean domain,  $E_n(R) = SL_n(R)$ .

Hence  $SK_1(R) = 0$ .

Pf. As in the proof for local rings, it suffices to show that every  $g \in GL_n(R)$  can be put in upper triangular form. If the first column of  $g$  contains an invertible element, we can conclude by induction.

$$g = \begin{pmatrix} \vdots \\ a_{i1} \\ \vdots \end{pmatrix} \quad * \quad \exists i, a_{i1} \neq 0$$

with  $\deg(a_{i1})$  is minimal.

If  $\deg(a_{i1}) = 0$ , then  $a_{i1} \in R^\times$  ( $1 = a_{i1}g + r$  with  $r = 0$ )

Otherwise,  $(a_{i1}) \neq R$ ,  $(a_{11}, a_{21}, \dots, a_{n1}) = R$ . So

$$\exists j \neq i \text{ s.t. } a_{j1} \notin (a_{i1}) \Rightarrow a_{j1} = a_{i1}q + r \text{ with } \deg(r) < \deg(a_{i1}), r \neq 0$$

replace row  $j$  by row  $j - q \cdot \text{row } i$

$$\text{new } a_{j1} \text{ is } a_{j1} - qa_{i1} = r$$

Since  $\deg(r) < \deg(a_{i1})$ , we can conclude by induction.  $\square$

Exmple.  $\exists$  PID's with  $SK_1 \neq 0$ . e.g.  $\mathbb{Z}[x][x^{-1}, (x^n - 1)^{-1} \forall n \geq 1]$

Number fields: For  $F/\mathbb{Q}$  a number field, we have

- Dirichlet's unit theorem:  $\mathcal{O}_F^\times = \mu_F \oplus \mathbb{Z}^{r_1 + r_2 - 1}$   
 where  $r_1 = \#$  real embeddings of  $F$   
 $r_2 = \#$  complex embeddings up to conjugates

• Thm (Bass-Milnor-Serre)  $SK_1(\mathcal{O}_F) = 0$

$$\Rightarrow K_1(\mathcal{O}_F) \xrightarrow[\det]{\cong} \mathcal{O}_F^\times = \mu_F \oplus \mathbb{Z}^{r_1 + r_2 - 1} \quad \left. \vphantom{\mathcal{O}_F^\times} \right\} \Rightarrow \mathbb{Z}_\ell K(\mathcal{O}_F) \longrightarrow \mathbb{P}_\ell \mathbb{Z}(\mathcal{O}_F)$$

is an equivalence of Picard groups etc.

Recall:  $K_0(\mathcal{O}_F) \xrightarrow[\text{rk, det}]{\cong} \mathbb{Z} \oplus \text{Pic}(\mathcal{O}_F)$

### $K_2$ & Milnor K-theory

Def. Let  $R$  be a ring and  $n \geq 3$ . The Steinberg group  $St_n(R)$  has

generators:  $x_{ij}(r)$ ,  $1 \leq i, j \leq n$ ,  $i \neq j$ ,  $r \in R$

relations:  $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \end{cases}$$

$$St(R) = \text{colim}_{n \rightarrow \infty} St_n(R)$$

$$\exists \text{ morphisms } St_n(R) \twoheadrightarrow E_n(R) \quad x_{ij}(r) \mapsto e_{ij}(r)$$

$$St(R) \twoheadrightarrow E(R)$$

Def (Milnor 67)  $K_2(R) = \ker(St(R) \twoheadrightarrow E(R)).$

Thm (Milnor)  $K_2(\mathbb{Z}) \cong \mathbb{Z}/2.$

Central extensions: If  $G$  is a group, a central extension of  $G$

$$\text{is a SES } 1 \rightarrow A \rightarrow \hat{G} \rightarrow G \rightarrow 1 \text{ where } A = Z(\hat{G}).$$

For any perfect group  $G$ , there exists a universal central extension

$$1 \rightarrow Z(\hat{G}_{\text{univ}}) \rightarrow \hat{G}_{\text{univ}} \rightarrow G \rightarrow 1$$

i.e.  $\forall$  central extension  $\hat{G} \rightarrow G, \exists! \hat{G}_{\text{univ}} \rightarrow \hat{G}$

Thm (Kervaire, Steinberg)

$$1 \rightarrow K_2(R) \rightarrow St(R) \rightarrow E(R) \rightarrow 1 \text{ is the universal central extension of } E(R).$$

Group homology interpretation  $H_*(G, \mathbb{Z}) = H_*(BG, \mathbb{Z})$  ↖ classifying space of  $G$

$$G^{\text{ab}} = H_1(G, \mathbb{Z}) \Rightarrow K_1(R) = H_1(GL(R), \mathbb{Z}).$$

For  $G$  perfect,  $Z(\hat{G}_{\text{univ}}) \cong H_2(G, \mathbb{Z}) \Rightarrow K_2(R) \cong H_2(E(R), \mathbb{Z}).$

Thm (Gorenstein)  $K_3(R) \cong H_3(St(R), \mathbb{Z}).$

Definition (Milnor K-theory) For  $F$  a field,

$$K_*^M(F) = T_{\mathbb{Z}}(F^\times) / (a \otimes (1-a), a \in F - \{0,1\})^{\text{2-sided ideal}}$$

↑ tensor algebra =  $\bigoplus_{n \geq 0} (F^\times)^{\otimes n}$

$$K_0^M(F) = \mathbb{Z} = K_0(F)$$

$$K_1^M(F) = F^\times = K_1(F)$$

Theorem (Matsumoto 69)  $K_2(F) \cong K_2^M(F)$

But  $K_n(F) \not\cong K_n^M(F)$  for  $n \geq 3.$