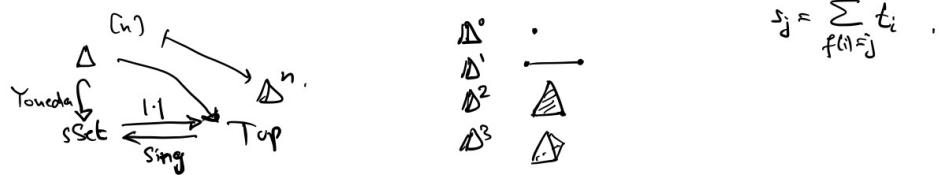


Geometric realization

$$\Delta \rightarrow \text{Top}, [n] \hookrightarrow \Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^n t_i = 1\}$$

$$[n] \xrightarrow{f} [m] \rightsquigarrow \Delta \rightarrow \Delta^m, (t_0, \dots, t_n) \mapsto (s_0, \dots, s_m)$$

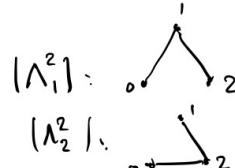


$\text{Sing}(\lambda)([n]) = \text{Hom}_{\text{Top}}(\Delta^n, X)$, 1:1 is left adjoint to Sing.
(universal property of the Yoneda embedding).

Example: $|\Delta^n| = \Delta^n$

$$\cdot |\partial\Delta^n| = \partial\Delta^n \quad \text{e.g. } \partial\Delta^0 = \emptyset \\ \partial\Delta^1 = \bullet \\ \partial\Delta^2 = \triangle$$

$$\cdot |\Lambda_k^n| = \partial\Delta^n \text{ with } \underbrace{\text{k-th face}}_{\substack{\text{opposite} \\ \text{k-th vertex}}} \text{ removed}$$



By the Yoneda lemma, every set X can be written as $X = \text{colim}_{\Delta^n \rightarrow X} \Delta^n$

$$\Rightarrow |X| = \text{colim}_{\Delta^n \rightarrow X} |\Delta^n| = X_n$$

Example: $\Delta^1/\partial\Delta^1$

$$\begin{array}{ccc} \partial\Delta^1 \hookrightarrow \Delta^1 & \xrightarrow{1:1} & \partial\Delta^1 \hookrightarrow \Delta^1 \\ \downarrow p_0 & & \downarrow \\ * \longrightarrow \Delta^1/\partial\Delta^1 & & * \longrightarrow |\Delta^1/\partial\Delta^1| \end{array}$$

$$\Rightarrow |\Delta^1/\partial\Delta^1| = \emptyset = S^1.$$

$$\text{More generally } |\Delta^n/\partial\Delta^n| = S^n.$$

Fact: $|X \times Y| = |X| \times |Y|$ if X or Y is finite (in general if the product is taken in compactly generated spaces)

Def: • A morphism $f: X \rightarrow Y$ in SSet is a weak equivalence if $|f|: |X| \rightarrow |Y|$ is a weak equivalence.
• $X \in \text{SSet}$, $x \in X_i$, $i \geq 0$, $\pi_i(X, x) = \pi_i(|X|, x)$.

Simplicial homotopy.

$X, Y \in \text{Set}$ $f, g: X \rightarrow Y$. A (simplicial) homotopy from f to g is $h: X \times \Delta^1 \rightarrow Y$

s.t.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \circ f \downarrow & h \nearrow & \downarrow \\ X \times \Delta^1 & \xrightarrow{h} & Y \\ \downarrow g & & \end{array}$$

$$\left(\Rightarrow |h|: |X| \times |\Delta^1| \rightarrow |Y| \right)$$

\Rightarrow a homotopy from $|f|$ to $|g|$

\triangle not transitive in general. (it is if Y is a Kan complex)

Def. $f: X \rightarrow Y$ is a homotopy equivalence if $\exists g: Y \rightarrow X$ s.t.
both $g \circ f$ and $f \circ g$ are homotopic to the identity.

Remark homotopy equivalence \Rightarrow weak equivalence

Nerve $N: \text{Cat} \rightarrow \text{sSet}$.

Def Let C be a small category. The nerve of C , $N(C) \in \text{sSet}$, is defined by

$$N(C)_n = \text{Hom}_{\text{Cat}}([n], C) = \left\{ \begin{array}{l} \text{composable sequences} \\ \text{of } n \text{ morphisms in } C \end{array} \right\}$$

\uparrow
 $\{0 \leq i < n\}$

$N(C)_0 = \text{objects of } C$

$N(C)_1 = \text{morphisms of } C$

$$\begin{array}{ccc} N(C)_0 & \xleftarrow{\substack{d_0 = \text{source} \\ id}} & N(C)_1 & \xleftarrow{\substack{d_0 \\ d_1}} & N(C)_2 \\ & \downarrow & & \downarrow & \\ & d_1 = \text{target} & & (f, g) & \end{array}$$

$\xrightarrow{f, g}$
 $d_0(f, g) = g$
 $d_1(f, g) = g \circ f$
 $d_2(f, g) = f$.

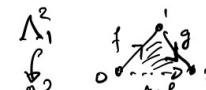
Exercise:

1) The nerve functor $N: \text{Cat} \rightarrow \text{sSet}$ is fully faithful.

2) A simplicial set X is in the cartesian square iff:

$\forall n \geq 0, \forall 0 \leq i \leq n$ and every map $\Delta_i^n \rightarrow X$,
then exists a unique extension $\Delta^n \rightarrow X$.

$$\begin{array}{ccc} \Delta_i^n & \xrightarrow{\Delta} & X \\ \downarrow & \nearrow \partial_i & \\ \Delta^n & & \end{array}$$

eg: Δ_1^2 

3) A simplicial set X is the nerve of a groupoid if:

$$\forall n \geq 0, \forall 0 \leq i \leq n, \quad \Delta_i^n \xrightarrow{\cong} X$$

Def. • A Kan complex is a simplicial set X s.t:

$\forall n \geq 0, \forall 0 \leq i \leq n$, every $\Lambda_i^n \rightarrow X$ can be extended

to $\Delta^n \rightarrow X$:

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\cong} & X \\ \downarrow & \nearrow & \nearrow \\ \Delta^n & \xrightarrow{\sim} & \end{array}$$

• A quasicategory is a simplicial set s.t: $\forall n \geq 0, \forall 0 \leq i < n$

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\cong} & X \\ \downarrow & \nearrow & \nearrow \\ \Delta^n & \xrightarrow{\sim} & \end{array}$$

Example: A simplicial group is a Kan complex.

Prop: 1) For every simplicial set X , there exists a Kan complex X' and a weak equivalence $X \rightarrow X'$.

2) A weak equivalence between Kan complexes is a homotopy equiv.

Cor: $s\text{Set}[\text{w.e.}] \simeq \text{Kan}[\text{weak equiv}]$.

Also: If X is a Kan complex, $x \in X_0$, $\text{Hom}_{\text{Set}_*}(\Delta^n / \partial \Delta^n, (X, x)) / \text{pointed homotopy} \simeq \pi_n(X, x)$.

Simplicial abelian groups

$A \in s\text{Ab}$, $C_* A = (A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\dots}) \in \text{Ch}_{\geq 0}(\text{Ab})$
(or any abelian category)

$$\partial_n = \sum_{i=0}^n (-1)^i d_i \quad \partial^2 = 0.$$

• $N_* A \in \text{Ch}_{\geq 0}(\text{Ab})$ normalized chain complex.

$$N_n A = \bigcap_{i=1}^n \ker(d_i) \quad \left(\begin{array}{l} N_* A \hookrightarrow C_* A \\ \cong \downarrow \\ C_* A / D_* A \end{array} \right)$$

degenerate simplices

• $N_* A \subset C_* A$ subcomplex

- Prop
- 1) $N_* A \hookrightarrow C_* A$ is a quasi-isomorphism
 - 2) There are natural isomorphisms $\pi_n(A, 0) \cong H_n(C_* A)$.

Thm (Dold-Kan correspondence)

$N : s\text{Ab} \longrightarrow \text{Ch}_{\geq 0}(\text{Ab})$ is an equivalence of categories.

Singular homology $X \in \text{Top}, A \in \text{Ab}$

$$C_*^{\text{sing}}(X, A) = C_*(\mathbb{Z}[\text{Sing}(X)] \otimes A)$$

$$H_*^{\text{sg}}(X, A) = H_*(C_*^{\text{sing}}(X, A))$$