REGENSBURG RESEARCH SEMINAR SS 2021

HERMITIAN K-THEORY OF RINGS

MARC HOYOIS AND DENIS NARDIN

Introduction

The goal of this seminar is to study some recent developments on the Hermitian K-theory of rings of integers following Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus, and Steimle [CDH+20]. A particular focus will be on the complete solution of Thomason's homotopy limit problem for number rings, which states that the Grothendieck–Witt spectrum of a number ring is 2-adically equivalent to the homotopy C_2-fixed points of its K-theory spectrum. In particular we will review the solution of this problem for fields using motivic homotopy techniques, following Bachmann and Hopkins [BH20].

1. **Introduction (27.04)**

Part I: The homotopy limit problem away from 2

The goal of the first half of the seminar will be to prove the following theorem:

Theorem 1 ([BH20, Theorem A.3])**.** *Let k be a field of characteristic not 2 and of finite virtual 2-cohomological dimension. Then the canonical map*

$$\mathrm{GW}(k) \to \mathrm{K}(k)^{hC_2}$$

is a 2-adic equivalence of spectra.

2. **Preliminaries on motivic spectra (04.05)**
   Define the homotopy t-structure, the slice filtration, and the effective homotopy t-structure on the ∞-category of motivic spectra [Mor03], [Bac17, Sections 2 and 3]. State Morel's $\mathbb{A}^1$-connectivity theorem [Mor03, Theorem 4.2.10]. Deduce that the homotopy t-structure and the effective homotopy t-structure over a field are complete, and describe their hearts (homotopy modules [Mor03, §5.2] and $\mathbb{A}^1$-invariant sheaves with framed transfers [BY20, Theorem 5.14], respectively). Recall Morel's fundamental computation $\underline{\pi}_0(\mathbf{1})_* \simeq \underline{\mathrm{K}}_*^{\mathrm{MW}}$.
3. **The motivic Hermitian K-theory spectrum (11.05)**
   Review the "motivic" features of the Hermitian K-theory of $\mathbb{Z}[1/2]$-schemes (the projective bundle formula, Nisnevich descent, $\mathbb{A}^1$-invariance on regular schemes), following Schlichting [Sch17]. Introduce the motivic spectra KGL, KQ, and KW representing (homotopy invariant) algebraic K-theory, Hermitian K-theory, and Witt theory, following for example [HJNY21, Section 6], [RØ16, Sections 3 and 4]. Explain the "Wood" and "Tate" cofiber sequences

$$\Sigma^{1,1}\mathrm{KQ} \xrightarrow{\eta} \mathrm{KQ} \to \mathrm{KGL}, \qquad \mathrm{KGL}_{hC_2} \to \mathrm{KQ} \to \mathrm{KW}.$$

4. **Voevodsky's slice filtration and Levine's coniveau filtration (18.05)**
   Define Levine's coniveau filtration for an $\mathbb{A}^1$-invariant presheaf of spectra $E\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$ [Lev08, Section 2]. Explain the comparison with Voevodsky's slice filtration for motivic

spectra over a perfect field [Lev08, Theorems 7.1.1 and 9.0.3], and the resulting computation of the slices of algebraic K-theory [Lev08, Theorem 6.4.2]:

$$s_n(\mathrm{KGL}) \simeq \Sigma^n_T \mathrm{H}\mathbb{Z}.$$

5. **The convergence of the slice filtration (25.05)**
Explain the relation between the slice filtration and the fundamental ideal filtration on $\underline{\pi}_0(\mathbf{1})_* \simeq \underline{\mathrm{K}}^{\mathrm{MW}}_*$ [Lev11]. Sketch the proof of the convergence theorem [BEØ20, Theorem 5.3], which is an extension of [Lev13, Theorem 7.3]. Deduce in particular that, if $2 \in k^\times$ and $\mathrm{vcd}_2(k) < \infty$, then a connective motivic spectrum over k is $(2, \rho)$-adically equivalent to its slice completion [BEØ20, Corollary 5.13(1)].
6. **The slices of Hermitian K-theory (01.06)**
Compute the slices of kq, and hence of $\mathrm{KQ} = \mathrm{kq}[\beta^{-1}]$ and $\mathrm{KW} = \mathrm{KQ}[\eta^{-1}]$, following [ARØ19, Theorem 3.2]. Using Milnor's conjecture on quadratic forms (which is in fact deduced from these computations in [RØ16]), compare the slice filtration of the Witt theory spectrum KW with the fundamental ideal filtration of the Witt group [RØ16, Lemma 6.4, Corollary 6.11].
7. **The homotopy limit problem away from 2 (08.06)**
Prove Theorem 1 following [BH20, Appendix A]. Explain how Theorem 1 generalizes to qcqs $\mathbb{Z}[1/2]$-schemes X with $\dim(X) < \infty$ and $\mathrm{vcd}_2(X) < \infty$ by Nisnevich descent [CM21, Corollary 3.27] and rigidity [BKSØ15, Theorem 3.6].

Part II: Grothendieck–Witt groups of rings of integers

The goal of the second half of the seminar is to study the Grothendieck–Witt theory of rings of integers in number fields, following [CDH+20]. In particular, we have the following version of the homotopy limit problem:

Theorem 2 ([CDH+20, Theorem 3.1.6])**.** *Let R be a Dedekind domain whose fraction field is a number field. Then the canonical map*

$$\mathrm{GW}^{\mathrm{s}}(R) \to \mathrm{K}(R)^{hC_2}$$

is a 2-adic equivalence of spectra.

The numbers below refer to [CDH+20].

8. **Algebraic surgery (15.06)**
Define m-quadratic Poincaré structures (1.1.2). Explain the procedure of surgery (1.1.13) and Lagrangian surgery (1.1.15). Define the L-groups with connectivity conditions (1.2.1) and use them to identify the L-groups of an m-quadratic Poincaré structure with the quadratic L-groups in a range (1.2.3, 1.2.8). Define the L-groups of short complexes (1.2.15) and prove that the non-negative genuine symmetric L-groups are isomorphic to the L-groups of short complexes (1.2.18)
9. **Regular coherent rings (22.06)**
Define r-symmetric Poincaré structures (1.1.2). Explain symmetric surgery (1.3.1,1.3.4) and use it to identify the symmetric L-groups for a duality compatible with the t-structure (1.3.3). Use this to identify the L-groups of regular coherent rings in a range (1.3.7, 1.3.8).
10. **Dévissage in Hermitian K-theory and the localization sequence (29.06)**
Prove the dévissage theorem for the symmetric Poincaré structure (2.1.8) and deduce from that the dévissage-localization sequence (2.1.9). Identify the boundary homomorphism for L-groups (2.2.1) with the classical residue homomorphism. If time allows, prove the localisation-completion square (2.1.12) and rigidity in quadratic L-theory (2.1.13).

11. **The homotopy limit problem for rings of integers (06.07)**
Explain the solution of the homotopy limit problem for finite fields of characteristic 2 (3.1.4), and how to use it to deduce the homotopy limit problem for Dedekind domains whose fraction field is a fraction field (3.1.6). Discuss the obstruction to an integral statement (3.1.8) and the L-theoretic obstruction to a general solution (3.1.11).

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