Hermitian K-theory of rings

Goal: Understand the Grothendieck–Witt theory of $\mathcal{O}_K$ for $[K:\mathbb{Q}]<\infty$.

In particular: $GW(\mathcal{O}_K)$ vs. $K(\mathcal{O}_K)^{hC_2}$ (Thomason's homotopy limit problem)

- A Poincaré ∞-category is a pair $(\mathcal{C}, Ϙ)$ where
  - $\mathcal{C}$ is a stable ∞-category
  - $Ϙ : \mathcal{C}^{op} \to Sp$ is a quadratic functor whose associated symmetric bilinear functor $B_Ϙ : \mathcal{C}^{op} \times \mathcal{C}^{op} \to Sp$ is perfect:

2-excisive

$$Ϙ(x \oplus y) = Ϙ(x) \oplus Ϙ(y) \oplus B_Ϙ(x,y)$$

$$B_Ϙ(x,y) = \hom_{\mathcal{C}}(x, D_Ϙ y) \quad , \quad D_Ϙ : \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{op}$$

$\rightsquigarrow$ Grothendieck–Witt spectrum $GW(\mathcal{C}, Ϙ)$

\+ fundamental fibre square:

$$\begin{tikzcd} GW(\mathcal{C},Ϙ) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L(\mathcal{C},Ϙ) \arrow[d] \\ K(\mathcal{C})^{hC_2} \arrow[r] & K(\mathcal{C})^{tC_2} \end{tikzcd}$$

$\mathcal{C} \in (\mathrm{Cat}_\infty^{ex})^{hC_2}$ & $K : \mathrm{Cat}_\infty^{ex} \to Sp$ is C_2-invariant.

<u>Main example</u>: $\mathcal{C} = \mathcal{D}^{perf}(R)$, R <u>commutative</u> ring.

$$B(P,Q) = \hom_R(P \otimes_R Q, R)$$
$$Ϙ^s(P) = B(P,P)^{hC_2} \quad \text{(symmetric forms on } P\text{)}$$
$$Ϙ^q(P) = B(P,P)_{hC_2} \quad \text{(quadratic forms on } P\text{)}$$

- Norm fibre sequence: $Ϙ^q(P) \xrightarrow{Nm} Ϙ^s(P) \longrightarrow B(P,P)^{tC_2} \simeq \hom_R(P, R^{tC_2})$
- We define more generally:

$$Ϙ^{\geq m} : \mathcal{D}^{perf}(R)^{op} \longrightarrow Sp \qquad m \in \mathbb{Z} \cup \{\pm\infty\}$$

$$\begin{tikzcd} Ϙ^{\geq m}(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \hom_R(P, \tau_{\geq m} R^{tC_2}) \arrow[d] \\ Ϙ^s(P) \arrow[r] & \hom_R(P, R^{tC_2}) \end{tikzcd}$$

$$\begin{array}{ccccccccccccc} Ϙ^q = Ϙ^{\geq +\infty} & \longrightarrow & \cdots & \longrightarrow & Ϙ^{\geq 2} & \longrightarrow & Ϙ^{\geq 1} & \longrightarrow & Ϙ^{\geq 0} & \longrightarrow & \cdots & \longrightarrow & Ϙ^{\geq -\infty} = Ϙ^s \\ & & & & \| & & \| & & \| & & & & \\ & & & & Ϙ^{gq} & & Ϙ^{ge} & & Ϙ^{gs} & & & & \end{array}$$

- If $2 \in R^\times$, all maps are isomorphisms.

Comparison with group completion.

$\mathrm{Proj}^{q/e/s}(R)$ = groupoid of non-degenerate quadratic/even/symmetric forms on R-modules.

$$\Longrightarrow \quad \Omega^\infty GW^{gq/ge/gs}(R) \simeq \left(\mathrm{Proj}^{q/e/s}(R), \oplus\right)^{gp}$$

Genuine equivariant perspective.

$\exists$ genuine C_2-spectrum $KR(\mathcal{C}, Ϙ)$ such that:

$$KR(\mathcal{C}, Ϙ)^e = K(\mathcal{C})$$
$$KR(\mathcal{C}, Ϙ)^{C_2} = GW(\mathcal{C}, Ϙ)$$
$$KR(\mathcal{C}, Ϙ)^{\phi C_2} = L(\mathcal{C}, Ϙ)$$

Thomason's "homotopy limit problem" (1982)

I hope this makes the idea of the proof clear. To get a real proof one just patches all the holes. [Thomason]

E genuine G-spectrum $\rightsquigarrow$ canonical map $E^G \longrightarrow E^{hG}$.

- Atiyah-Segal: $K^*_G(X) \longrightarrow K^*(X_{hG})$ is completion with respect to the augmentation ideal $I_G \subset K^0_G(*) = R_{\mathbb{C}}(G)$.
- Löffler, Greenlees-May: $MU^*_G(X) \longrightarrow MU^*(X_{hG})$ is I_G-adic completion.
- The Segal conjecture (Carlsson): $\mathbb{S}^G \longrightarrow \mathbb{S}^{hG}$ is I_G-adic completion.
- Thomason: L/F Galois extension with Galois group G, $\ell \in F^\times$.

$$K(F)/\ell^n[\beta^{-1}] \xrightarrow{\sim} K(L)^{hG}/\ell^n[\beta^{-1}].$$

- For $E = KR(\mathcal{C}, \text{Ϙ})$: how close is $GW(\mathcal{C}, \text{Ϙ}) \longrightarrow K(\mathcal{C})^{hC_2}$ (equivalently, $L(\mathcal{C}, \text{Ϙ}) \longrightarrow K(\mathcal{C})^{tC_2}$) to an equivalence?

Results

Theorem A (Hu-Kriz-Ormsby 2011, Berrick-Karoubi-Schlichting-Østvaer 2015)

Let X be a qcqs scheme of finite Krull dimension with $2 \in \mathcal{O}(X)^\times$ and $\mathrm{vcd}_2(\kappa(x)) < \infty$ for all $x \in X$. Then the map

$$GW(X) \longrightarrow K(X)^{hC_2}$$

is a 2-adic equivalence.

It is even an isomorphism if X has no formally real residue fields.

Theorem B (Calmès-Dotto-Harpaz-Hebestreit-Land-Moi-Nardin-Nikolaus-Steimle 2020)

Let R be a Dedekind ring whose fraction field is a number field. Then the map

$$GW^s(R) \longrightarrow K(R)^{hC_2}$$

is a 2-adic equivalence.

Motivic homotopy theory

The proof of Theorem A uses stable motivic homotopy theory.

Fix a qcqs base scheme S.

Let Sm_S be the category of smooth S-schemes of finite presentation.

Definition (Morel - Voevodsky)

A motivic space over S is a presheaf $F : Sm_S^{op} \longrightarrow \text{Spaces}$ satisfying:

- Nisnevich descent: if $V \xrightarrow{p} X$ is étale & $p^{-1}(Z) \xrightarrow{\sim} Z$, then

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(X \setminus Z) \arrow[d] \\ F(V) \arrow[r] & F(V - p^{-1}(Z)) \end{tikzcd}$$

- $\mathbb{A}^1$-invariance: $F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$.

Remark: Nisnevich descent $\Longleftrightarrow$ descent for the Nisnevich topology, whose points are henselian local rings.

Localisation functors:

$$\begin{tikzcd}
\mathcal{P}(Sm_S) \arrow[r, shift left, "L_{Nis}"] \arrow[d, shift right, "L_{\mathbb{A}^1}"'] \arrow[dr, "L_{mot}"] & Shv_{Nis}(Sm_S) \arrow[l, hook, shift left] \\
\mathcal{P}_{\mathbb{A}^1}(Sm_S) \arrow[u, hook, shift right] & \mathcal{H}(S) \arrow[l, hook] \arrow[u, hook]
\end{tikzcd}$$

Basic facts

- L_{Nis} is left exact
- $L_{\mathbb{A}^1}(F) \simeq \operatorname{colim}_{n \in \Delta^{op}} F(- \times \mathbb{A}^n)$ $\Longrightarrow$ $L_{\mathbb{A}^1}$ & L_{mot} preserve finite products

  $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[t_0, \dots, t_n] / (\sum_i t_i - 1)$

- Let $S^{p,q} = S^{p-q} \wedge (\mathbb{G}_m, 1)^{\wedge q}$. In $\mathcal{H}(S)$, we have:

$$S^{2,1} \simeq (\mathbb{P}^1, \infty) \simeq \mathbb{A}^1 / \mathbb{A}^1 - 0 =: T$$

$$S^{2n,n} \simeq \mathbb{A}^n / \mathbb{A}^n - 0$$

$$S^{2n-1,n} \simeq \mathbb{A}^n - 0 .$$

Motivic spectra.

- A motivic S^1-spectrum over S is a presheaf of spectra $E : Sm_S^{op} \longrightarrow Sp$ satisfying Nisnevich descent & $\mathbb{A}^1$-invariance.

  Notation: $SH^{S^1}(S)$.

- A motivic spectrum over S is a $\mathbb{P}^1$-spectrum in $\mathcal{H}(S)_*$:

$$(F_0, F_1, F_2, \dots), \quad F_i \in \mathcal{H}(S)_*, \qquad F_i \simeq \Omega_{\mathbb{P}^1} F_{i+1}.$$

  Equivalently, since $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$, it is a $\mathbb{G}_m$-spectrum in $SH^{S^1}(S)$:

$$(E_0, E_1, E_2, \dots), \quad E_i \in SH^{S^1}(S), \quad E_i \simeq \Omega_{\mathbb{G}_m} E_{i+1}.$$

  Notation: $SH(S)$.

$$\begin{tikzcd} \mathcal{H}(S)_* & SH^{S^1}(S) \arrow[l, "\Omega^\infty"'] & SH(S) \arrow[l, "\Omega^\infty_{\mathbb{G}_m}"'] \arrow[ll, bend left, "\Omega^\infty_{\mathbb{P}^1}"] \end{tikzcd}$$

with left adjoints $\Sigma^\infty, \Sigma^\infty_{\mathbb{G}_m}, \Sigma^\infty_{\mathbb{P}^1}$.

Examples

- The motivic sphere spectrum is the unit object $\mathbb{1} \in SH(S)$.
- The motivic Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ or $M\mathbb{Z} \in SH(S)$ represents motivic cohomology: for $X \in Sm_S$ and $p, q \in \mathbb{Z}$,

$$H\mathbb{Z}^{p,q}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} H\mathbb{Z}] \simeq H^p_{mot}(X, \mathbb{Z}(q))$$

- The motivic K-theory spectrum $KGL \in SH(S)$ is

$$KGL = (K, K, \dots) \quad \text{with} \quad \Omega_{\mathbb{P}^1} K \simeq K \quad \text{(projective bundle formula)}.$$

$$\hom(\Sigma^\infty_{\mathbb{P}^1} X_+, KGL) \simeq KH(X) \qquad (KH = L_{\mathbb{A}^1} K)$$

$2 \in \mathcal{O}(S)^\times$ • The motivic Hermitian K-theory spectrum KQ or $KO \in SH(S)$ is

$$KQ = (GW, GW, \dots) \quad \text{with} \quad \Omega^4_{\mathbb{P}^1} GW \simeq GW \qquad \left(\begin{array}{l} \Omega_{\mathbb{P}^1} GW \simeq GW^{[-1]} \\ \& \ GW^{[-4]} \simeq GW \end{array} \right)$$

$$\hom(\Sigma^\infty_{\mathbb{P}^1} X_+, KQ) \simeq (L_{\mathbb{A}^1} GW)(X)$$

<u>Remark</u> If X is regular, $KH(X) = K(X)$ and $(L_{\mathbb{A}^1} GW)(X) = GW(X)$.

The slice filtration.

- $SH(S)^{\mathrm{eff}} \subset SH(S)$ stable subcategory generated under colimits by $\Sigma^\infty_{\mathbb{P}^1} X_+$.

$$\cdots \subset \Sigma^{q+1}_{\mathbb{P}^1} SH(S)^{\mathrm{eff}} \subset \Sigma^{q}_{\mathbb{P}^1} SH(S)^{\mathrm{eff}} \subset \cdots \quad \subset SH(S)$$

- Let $f_q : SH(S) \longrightarrow \Sigma^q_{\mathbb{P}^1} SH(S)^{\mathrm{eff}}$ be the right adjoint to the inclusion.

$\rightsquigarrow$ for any $E \in SH(S)$, we have a filtration

$$\cdots \to f_{q+1}E \to f_q E \to f_{q-1}E \to \cdots \to E$$

called the slice filtration of E. The slices of E are

$$s_q E = \mathrm{cofib}(f_{q+1}E \to f_q E).$$

Applying $[\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{*,t}(-)]$, we get the weight t slice spectral sequence:

$$(s_q E)^{s,t}(X) \Longrightarrow E^{s,t}(X).$$

Examples (over a field k)

- $s_q H\mathbb{Z} = \begin{cases} H\mathbb{Z} & \text{if } q=0 \\ 0 & \text{otherwise,} \end{cases}$
- $s_0 \mathbb{1} \simeq H\mathbb{Z}$ ($\Rightarrow$ the slices $s_q E$ are always $H\mathbb{Z}$-modules)
  a.k.a. motives
- $s_q KGL \simeq \Sigma^{2q,q} H\mathbb{Z}$

Voevodsky, Levine

In this case the slice SS is the motivic Atiyah-Hirzebruch SS:

$$H^p_{mot}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

- If $\mathrm{char}(k)=0$, the slices $s_q \mathbb{1}$ are known (Levine). If $k=\bar{k}$:

weight 0 slice SS for $\mathbb{1}$ $\equiv$ the Adams-Novikov SS

$$H^s(\mathcal{M}_{fg}, \omega^{\otimes t}) = \mathrm{Ext}^{s,2t}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_{2t-s}(\mathbb{S})$$

- The slices of KQ are known (Röndigs–Østvær)

Convergence of the slice filtration.

For any $E \in SH(S)$, $E \simeq \operatorname{colim}_{q \to -\infty} f_q E$. However, $\lim_{q \to \infty} f_q E \not\simeq 0$ in general.

Example: $n \in \mathcal{O}(S)^\times \Rightarrow \exists\ H\mathbb{Z}/n_{et} \in SH(S)$ s.t. $H\mathbb{Z}/n_{et}^{p,q}(X) \simeq H^p_{et}(X, \mu_n^{\otimes q})$.

Then $H\mathbb{Z}/n_{et}$ is effective and τ-periodic for $\tau \in H\mathbb{Z}/n_{et}^{0,\varphi(n)}(S)$.

$$\Rightarrow H\mathbb{Z}/n_{et} \in \bigcap_{q \geq 0} \Sigma^q_{\mathbb{P}^1} SH(S)^{eff} \Rightarrow s_* H\mathbb{Z}/n_{et} = 0.$$

- At best, the slice SS gives us information about the slice completion:

$$sc(E) = \lim_{n \to \infty} \operatorname{cofib}(f_n E \to E)$$

Example: KGL is slice-complete:

$\exists$ non-degenerate t-structure on $SH(k)$ s.t. $f_q KGL \in SH(k)_{\geq q} \Rightarrow \lim_{q \to \infty} f_q KGL = 0$

$\Rightarrow$ the motivic AHSS converges strongly.

Convergence of the slice filtration (continued)

- The Hopf fibration
$$\begin{array}{ccc} \mathbb{A}^2 - 0 & \xrightarrow{\eta} & \mathbb{P}^1 \\ \wr\!\wr & & \wr\!\wr \\ S^{3,2} & & S^{2,1} \end{array}$$
defines an element
$$\begin{array}{ccc} \eta & \in & \pi_{1,1}(\mathbb{1}) \\ \downarrow & & \downarrow \\ 0 & \in & \pi_{1,1}(H\mathbb{Z}) = 0 \end{array}$$

$\Rightarrow$ effective slice-complete spectra are η-complete.

- $\mathbb{1}$ & KQ are <u>not</u> η-complete [unlike in topology, η is not nilpotent!]

But we have
$$sc(\mathbb{1}) = \mathbb{1}^\wedge_\eta$$
$$sc(kq) = kq^\wedge_\eta \qquad (kq = \text{effective version of } KQ)$$

Strategy of the proof of Theorem A

- (Heard) : over any scheme S, $KQ^{\wedge}_{\eta} \xrightarrow{\simeq} KGL^{hC_2}$.

  This uses the Wood cofibre sequence $\Sigma^{1,1} KQ \xrightarrow{\eta} KQ \longrightarrow KGL$

- (Bachmann – Hopkins) : if $\mathrm{char}(k) \neq 2$ and $\mathrm{vcd}_2(k) < \infty$, $KQ/2 \xrightarrow{\simeq} KQ^{\wedge}_{\eta}/2$

  This uses the computation of $s_* KQ$ and a general convergence theorem :

> **Theorem** (Levine, Bachmann-Elmanto-Østvær)
> Let k be a field, $\ell \in k^{\times}$ such that $\mathrm{vcd}_{\ell}(k) < \infty$.
> If $E \in SH(k)_{\geq c}$ for some $c \in \mathbb{Z}$, then $sc(E) \simeq E^{\wedge}_{(\ell,\rho)}$

Here, $\rho = [-1] \in \pi_{-1,-1}(\mathbb{1})$ is the element induced by $\mathrm{Spec}\,\mathbb{Z} \xrightarrow{-1} \mathbb{G}_m$.

- The result for general X follows using that $GW/2$ and $K/2$ are both rigid Nisnevich sheaves (Gabber, Knebusch)

  $F(\mathcal{O}^h_{X,x}) \xrightarrow{\simeq} F(\kappa(x))$

Strategy of the proof of Theorem B

Theorem (Localisation - dévissage)
Let R be a Dedekind ring, $X = \operatorname{Spec} R$, $Z \subset X$ a finite closed subset, $U = X \setminus Z$. For any $m \in \mathbb{Z}$, we have a fibre sequence

$$\bigoplus_{x \in Z} GW(\kappa(x), (\text{Ϙ}^{s})^{[m-1]}) \longrightarrow GW(X, (\text{Ϙ}^{s})^{[m]}) \longrightarrow GW(U, (\text{Ϙ}^{s})^{[m]})$$

via $P \mapsto P^{*}[m]$

Theorem Let $q = 2^n$ and $m \in \mathbb{Z}$. Then $GW(\mathbb{F}_q, (\text{Ϙ}^{s})^{[m]}) \simeq K(\mathbb{F}_q)^{hC_2}$.