the homotopy t-structures

Recall: C stable

then a t-structure on $\mathcal{C}$ is given by full subcats

$$(C_{\leq 0}, C_{\geq 0})$$

s.t.

i) $X \in \mathcal{C}_{\geq 0}$ $\quad Y \in C_{\leq 0}$

then

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(X, \Omega Y) = 0$$

ii) $\Sigma C_{\geq 0} =: C_{\geq 1} \subseteq C_{\geq 0} \qquad C_{\leq -1} \subseteq C_{\leq 0}$

iii) $X \in C$ then there exists

$$\begin{array}{ccccc} X_{\geq 0} & \longrightarrow & X & \longrightarrow & X_{\leq -1} \\ \cap & & & & \cap \\ \mathcal{C}_{\geq 0} & & & & \mathcal{C}_{\leq -1} \end{array}$$

exact.

Rem: there are localization functors

$$\tau_{\leq n} : C \underset{\supseteq}{\rightleftarrows} C_{\leq n} \qquad \mathcal{C}_{\geq n} \underset{\subseteq}{\leftrightarrows} \mathcal{C} : \tau_{\geq n}$$

Goal for today:

Thm k field

let $SH(k)_{\geq 0} \subseteq SH(k)$ smallest full subcat s.t.

i) $\Sigma^{p,q}_{u} \Sigma^{\infty}_{\mathbb{P}^1} X \in SH(k)_{\geq 0}$

$S^{p-q} \wedge \mathbb{G}_m^{q}$

i) stable under extensions

ii) stable under colimits

then this determines (gives the connective part) of a t-structure on $SH(k)$ called the homotopy t-structure

Furthermore we have

- $E \in SH(k)_{\geq d} \iff \underline{\pi}_{p,q}(E) = 0$ if $p-q < d$

  $E \in SH(k)_{\leq d} \iff \underline{\pi}_{p,q}(E) = 0$ if $p-q > d$

- the t-structure is left and right complete

left: $SH(k) = \lim_{\mathbb{Z}} (\cdots \to SH(k)_{\leq 1} \xrightarrow{\tau_{\leq 0}} SH(k)_{\leq 0} \to \cdots)$

right: $SH(k) = \lim_{\mathbb{Z}} (\cdots \to SH(k)_{\geq 0} \xrightarrow{\tau_{\geq 1}} SH(k)_{\geq 1} \to \cdots)$

- $SH(k)^{\heartsuit} = SH(k)_{\geq 0} \cap SH(k)_{\leq 0}$ = "homotopy modules"

  that is

  $(A_0, A_1, \cdots)$ $\quad A_i \cong \Omega_{\mathbb{G}_m} A_{i+1}$

  w/ A_i strictly $\mathbb{A}^1$-invariant sheaf of abelian groups

  ($H^n_{Nis}(-, A_i)$ is $\mathbb{A}^1$-invariant $\forall n$)

Cor: If k is perfect

$$SH^{eff}(k)_{\geq 0} = \langle \Sigma^{\infty}_{\mathbb{P}^1} X_+ \rangle_{\substack{colim \\ extensions}}$$

determines a t-structure on $SH^{eff}(k)$

w/ $SH(k)^{\mathrm{eff},\heartsuit}$ = strictly $\mathbb{A}^1$-invariant abelian sheaves w/ framed transfers (admit extension to $h\,\mathrm{Corr}^{fr}(k)$)

Remark: $SH(S)_{\geq 0}$ determines a t-structure over any base [HA, 1.4.4.11]

Prop: S of finite Krull dimension then $Shv_{Sp}(S)$ admits a left and right complete t-structure with

$$Shv_{Sp}(S)_{\geq n} = \{E \in Shv_{Sp}(S) \mid \pi_m(E) = 0 \ \forall m < n\}$$

$$Shv_{Sp}(S)_{\leq n} = \{E \in Shv_{Sp}(S) \mid \pi_m(E) = 0 \ \forall m > n\}$$

Proof: Combine [SAG, 1.3.3.7] + Postnikov completeness of $Sh(S)$

□

Def: there is a localization functor

$$L_{\mathbb{A}^1} : Shv_{Sp}(S) \to SH^{S^1}(S)$$

which we can compute

$$L_{\mathbb{A}^1} E = \operatorname*{colim}_{n \in \Delta^{op}} E(- \times \mathbb{A}^n)$$

Def: We say S has the connectivity property if:

$E \in Shv_{Sp}(S)$ is connective

$\Rightarrow$ $L_{\mathbb{A}^1} E$ is connective

Thm (Morel)

Fields have the connectivity property

Prop: Assume S has the connectivity property. Then the t-structure on $Sh_{Sp}(S)$ restricts to $SH^{S^1}(S)$ (including the truncation functors). This t-structure is called <u>the homotopy t-structure</u> is left and right complete and we have

$$E \in SH^{S^1}(S)_{\geq n} \Leftrightarrow \pi_m(E) = 0 \quad \forall m < n$$

$$E \in SH^{S^1}(S)_{\leq n} \Leftrightarrow \pi_m(E) = 0 \quad \forall m > n$$

Proof: It suffices to check

$$\tau_{\geq n} : Sh_{Sp}(S) \to Sh_{Sp}(S)_{\geq n}$$

preserves $\mathbb{A}^1$-invariance.
Since $L_{\mathbb{A}^1} \tau_{\geq n} E$ is n-connective we get

$$\begin{tikzcd}
\tau_{\geq n} E \arrow[r, "id"] \arrow[d] & \tau_{\geq n} E \arrow[r] & E \arrow[d, "\wr"] \\
L_{\mathbb{A}^1} \tau_{\geq n} E \arrow[ur, dashed] \arrow[rr] & & L_{\mathbb{A}^1} E
\end{tikzcd}$$

$\to \tau_{\geq n} E$ is a direct summand of a $\mathbb{A}^1$-invar. sheaf.

$\square$

Cor: $SH^{S^1}(S)^{\heartsuit} = Ab^{stc}(S)$ = "strictly $\mathbb{A}^1$-inv. sheaves of ab. groups"

Proof:

$$Shsp(S)^{\heartsuit} = \lim \Big(\cdots \to EM_2(S) \xrightarrow{\Omega} EM_1(S) \xrightarrow{\Omega} EM_0(S) \Big)$$

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "\pi_3", "\simeq"'] & EM_2(S) \arrow[r, "\Omega"] \arrow[d, "\pi_2", "\simeq"'] & EM_1(S) \arrow[r, "\Omega"] \arrow[d, "\pi_1", "\simeq"'] & EM_0(S) \arrow[d, "\pi_0", "\simeq"'] \\
\cdots \arrow[r] Ab(S) & Ab(S) \arrow[r] & Group(S) \arrow[r] & Set(S)
\end{tikzcd}$$

$$\lim \Big(\cdots \to Ab(S) \xrightarrow{id} Ab(S) \to Group(S) \to Set(S) \Big)$$

so we have to identify those $A \in Ab(S)$ s.t. $K(A,n)$ is $\mathbb{A}^1$-local for all n but

$$\pi_0 Map_{Shsp(S)}(\Sigma^\infty X_+, \Sigma^n HA) = H^n_{Nis}(X, A)$$

$\square$

lemma: k a field $E \in SH^{S^1}(k)$ then

$$\pi_n(\Omega_{\mathbb{G}_m} E) \to \Omega_{\mathbb{G}_m} \pi_n(E) \quad \text{is an iso}$$

$$\begin{tikzcd}
 & \Omega_{\mathbb{G}_m}\pi_n(E) \arrow[d, phantom, "\cong"] \\
\ker \arrow[r] \arrow[d] & \pi_n(E)(- \times \mathbb{G}_m) \arrow[d, shift left] \\
0 \arrow[r] & \pi_n(E)(- \times k) \arrow[u, shift left]
\end{tikzcd}$$

Cor: We have isos in $SH^{S^1}(k)$

$$\Omega_{\mathbb{G}_m}\tau_{\geq n}E \to \tau_{\geq n}\Omega_{\mathbb{G}_m}E \quad \text{and}$$

$$\tau_{\leq n}\Omega_{\mathbb{G}_m}E \to \Omega_{\mathbb{G}_m}\tau_{\leq n}E$$

Prop: k a field then pointwise truncation determines a t-structure on

$$SH(k) = \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{G}_m}} SH^{S^1}(k) \xrightarrow{\Omega_{\mathbb{G}_m}} SH^{S^1}(k) \right)$$

that is

$$E \in SH(k)_{\geq d} \iff \Omega^{\infty+n}_{\mathbb{G}_m} E \in SH^{S^1}(S)_{\geq d} \quad \forall n \in \mathbb{Z}$$

$$E \in SH(k)_{\leq d} \iff \Omega^{\infty+n}_{\mathbb{G}_m} E \in SH^{S^1}(S)_{\leq d} \quad \forall n \in \mathbb{Z}$$

In particular we have

- the t-structure is left and right complete
- $E \in SH(k)_{\geq d} \iff \underline{\pi}_{p,q}(E) = 0$ for $p-q < d$

  $E \in SH(k)_{\leq d} \iff \underline{\pi}_{p,q}(E) = 0$ for $p-q > d$
- $SH(k)^{\heartsuit}$ = "homotopy modules"

  $(A_0, A_1, \ldots)$ $A_i \simeq \Omega_{\mathbb{G}_m} A_{i+1}$

  $A_i \in Ab^{stc}(S)$

Milnor-Witt k-theory

Def: k field

Then the Milnor-Witt-k-theory ring of k

$$K_*^{MW}(k)$$

is the graded associative ring generated by symbols

- $[u]$ in degree 1 for any $u \in k^\times$
- η in degree -1

sat. the following relations

1. (Steinberg)
   $a \in k^\times \setminus \{1\}$ then
   $$[a] \cdot [1-a] = 0$$

2. $(a, b) \in (k^\times)^2$
   $$[ab] = [a] + [b] + \eta \cdot [a] \cdot [b]$$

3. $u \in F^\times$
   $$[u] \cdot \eta = \eta \cdot [u]$$

4. set $H := \eta \cdot [1] + 2$ then
   $$\eta \cdot H = 0$$

Rem: $K_*^{MW}(k)/\eta = K_*^M(k)$ Milnor K-theory

$K_*^{MW}(k)/h = K_*^W(k)$ Witt K-theory

$K_0^{MW}(k) = GW_0(k)$

„Element" $\mapsto$ „Matrix"

Thm: there exists a homotopy module

$$(\underline{K}_0^{MW}, \underline{K}_1^{MW}, \dots)$$

w/ $\underline{K}_*^{MW}(k) = K_*^{MW}(k)$ for k a field

and an iso

$$\underline{\pi}_{-n,-n}(\mathbb{1}) \cong \underline{K}_n^{MW}$$

where the Hopf fibration

$$\begin{array}{ccc} \mathbb{A}^2 \setminus \{0\} & \longrightarrow & \mathbb{P}^1 \\ \cong & & \cong \\ S^{3,2} & & S^{2,1} \end{array}$$

corresponds to η