The motivic Hermitian K-theory spectrum

I. KQ and W for affine schemes

II. GW in very general settings and what one can prove about it

III. Motivic representability of these functors.

I. Hermitian K-theory and Witt theory of rings.

$$\underset{\text{comm. ring}}{R} \rightsquigarrow \mathrm{Proj}^q(R) := \left\{ (V, \varphi) \;\middle|\; \begin{array}{l} V - \text{f.g. proj } R\text{-module} \\ \varphi : V \otimes_R V \to R \\ \text{non-degenerate} \\ \text{\& symmetric} \end{array} \right\}$$

- it is symmetric monoidal w.r.to $\perp$

$\rightsquigarrow$ its group completion is the space $K^h(R)$ of (connective part of) hermitian K-theory

$$KQ_n(R) := \pi_n K^h(R), \quad n \geq 0$$

and using deloopings one might define $KQ_n(R), n<0$.

Since 2 is invertible, (V, φ) is a direct summand of a hyperbolic form (e.g. $(R, a) \perp (R, -a) \simeq (R^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$),

One gets that $KQ_n(R) \simeq \pi_n(BO(R)^+), \quad n \geq 1$

where $O(R) := \underset{\perp, n}{\mathrm{colim}}\, \mathrm{Aut}(\mathbb{H}^{\oplus n})$

$\mathbb{H} := (R \oplus R, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$.

Turns out that the negative Hermitian K-theory has a nice algebraic description — in terms of Witt groups.

For a comm. ring $R \rightsquigarrow W(R) := \dfrac{\mathrm{Ob}\,\mathrm{Proj}^{\varphi}(R)/\mathrm{isom}}{\langle \text{metabolic spaces} \rangle}$

where (V, φ) is metabolic,

if $\exists L \subset V$ s.t. $\varphi|_L = 0$ & $L \simeq L^{\perp}$.

$W(R)$ is an abelian group, $W(k)$ is the Witt ring of anisotropic quadr. forms$/k$ (char $k \neq 2$).

Balmer has introduced triangular Witt groups, i.e. given a triangulated category $\mathcal{T}$ with duality, $\delta \in \{\pm 1\}$

$\#$, can

$\#: \mathcal{T}^{op} \to \mathcal{T}$

can : $Id \xrightarrow{\sim} \# \circ \#^{op}$

consistent with Δ-str. ← depends on δ

symmetric space (P, φ): $\varphi : P \xrightarrow{\sim} P^{\#}$ s.t.

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\wr\, can"'] & P^{\#} \\ P^{\#\#} \arrow[ur, "\varphi^{\#}"'] & \end{tikzcd}$$

metabolic symmetric space (P, φ): $L \xrightarrow{\alpha} P \xrightarrow{\alpha^{\#}\varphi} L^{\#} \xrightarrow{+1}$

$L^{\#} \to L[1]$ "is δ-symmetric"

$$W(\mathcal{T}, \ldots) := \frac{\text{symm. spaces}}{\text{metabolic spaces}}$$

and $W\left(D^b(\mathrm{Proj}\,R), \# = R\mathrm{Hom}(-, R), \mathrm{can}, \delta = 1\right) \simeq W(R)$.

↑ symmetric forms

There is a new feature of Witt groups when one considers triangulated categories with duality:

$\mathcal{T} := (\mathcal{T}, \#, \mathrm{can}, \delta)$, $n \in \mathbb{Z}$

$\rightsquigarrow$

$\mathcal{T}^{[n]} := \left(\mathcal{T}, [n] \circ \#, (-1)^{\frac{n(n+1)}{2}+n} \cdot \mathrm{can}, (-1)^n \delta\right)$ is also a triang. cat with duality $\rightsquigarrow W^n(\mathcal{T}) := W(\mathcal{T}^{[n]})$

Note the change of signs! It is related to the fact that if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is exact triangle, then $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y$ is an exact triangle.

So, in fact, $\mathcal{T}^{[2]} \simeq (\mathcal{T}, \#, \mathrm{can}, -\delta)$

$\mathcal{T}^{[4]} \simeq \mathcal{T}$.

$W^n(\mathcal{T})$ is 4-periodic

If $\mathcal{T} = (\mathrm{Proj}(R), \#, \mathrm{can}, 1)$, then $W(\mathcal{T}^{[2]})$ is the Witt group of skew-symmetric bilinear forms.

[We could consider also symplectic K-theory of a ring replacing symmetric forms by skew-symmetric, hence $O(R)$ by $Sp(R)$]

II. Grothendieck-Witt spectrum.

k - base ring, e.g. $\mathbb{Z}[1/2]$.

When 2 is invertible, there are equivalent constructions

(Schlichting) $\quad dg\,CatWD_k \xrightarrow{GW} Sp$

1-category of small dg-categories with w.e. and duality

$(\mathcal{A}, w, \vee, can)$

$w \subset Z^0\mathcal{A}^{ptr}$, $\vee : \mathcal{A}^{op} \to \mathcal{A}$, $can : 1\!\!1 \to \vee \circ \vee^{op}$

weak equivalences and $\mathcal{A}^w$ is closed under $\vee$ — objectwise-weak equivalence

(CDHHLMNNS) $\quad Cat_\infty^p \xrightarrow{GW} Sp$

in part. Part II

∞-category of Poincaré ∞-categories

$\left(\mathcal{C},\ \underset{\text{-quadratic}}{Ϙ} : \mathcal{C}^{op} \to Sp \right)$ that induces perfect duality

For a scheme X one can consider

- $(sPerf(X), quis, \#, can)$

$\#_{\mathcal{O}_X}(E) := RHom_{\mathcal{O}_X}(E, \mathcal{O}_X)$

or $(D^{perf}(R), Ϙ)$ $\quad Ϙ(M) := \hom(M \otimes M, R)^{hC_2}_{hC_2}$

Similar to triangulated categories with duality there are certain shifts possible: $\mathcal{A} \rightsquigarrow \mathcal{A}^{[n]}$

and $\mathcal{A}^{[n+4]} \simeq \mathcal{A}^{[n]}$

using the monoidal structure one can describe $\mathcal{A}^{[n]}$ as $\mathcal{A} \otimes \mathrm{Perf}(k)^{[n]}$

and $\mathrm{Perf}(k)^{[n]}$ is $\mathrm{Perf}(k)$ where the duality is given by $\mathrm{Hom}(-, k[n])$.

$$GW(\mathcal{A}^{[n]}) =: GW^{[n]}(\mathcal{A}) \text{ is 4-periodic}$$

For $\mathcal{A} = \mathrm{Perf}(R)$ these constructions are related to group completion:

$$\Omega^\infty GW(\mathrm{Spec}\, R)^\circ \simeq BO(R)^+$$

and $\Omega^\infty GW^{[2]}(\mathrm{Spec}\, R)^\circ \simeq B\,Sp(R)^+$.

Moreover, GW is "symmetric monoidal" which makes $GW_i(X) := \pi_i GW(X)$ into a graded ring.

$$\left(\mathrm{Perf}(X) \otimes \mathrm{Perf}(Y) \simeq \mathrm{Perf}(X \times Y) \right)$$

Properties of GW:

- Localization theorem:

underlying triang./stable ∞-cat's [Schlichting] Morita-exact

if $\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2$ is "exact",

then $GW(\mathcal{A}_0) \to GW(\mathcal{A}_1) \to GW(\mathcal{A}_2)$ is exact

Recall that Morita-exact means that $h\mathcal{A}_0 \underset{\text{fully faithful}}{\subseteq} h\mathcal{A}_1$

and $h\mathcal{A}_1/h\mathcal{A}_0 \longrightarrow h\mathcal{A}_2$ is cofinal = $\begin{cases} \text{fully faithful} + \\ \text{every object of } h\mathcal{A}_2 \text{ is a direct} \\ \text{summand of Ob } h\mathcal{A}_1/h\mathcal{A}_0 \end{cases}$

Example: $\mathcal{A} \longrightarrow Fun([1], \mathcal{A}) \longrightarrow \underbrace{\mathcal{A}^{[1]}}_{\text{has "shifted" duality}}$

$A \longmapsto (A \xrightarrow{id} A)$

$(B \xrightarrow{f} C) \longmapsto cone(f)$

is exact

hence $GW(\mathcal{A}) \to GW(Fun[1], \mathcal{A}) \to GW(\mathcal{A}^{[1]})$ (*)

$\quad = GW^{[1]}(\mathcal{A})$

$\delta \downarrow +1$ is exact

$GW(\mathcal{A})$

Moreover, one can check that $\delta = \eta \cup -$

where $\eta \in GW^{[-1]}_{-1}(k)$

$\cong$

(Hopf map $\in \pi_{-1,-1}(k)$)

$$\mathrm{Fun}([1], \mathcal{A}) \rightleftarrows \underbrace{\mathcal{H}\mathcal{A} := \mathcal{A} \times \mathcal{A}^{op}}_{\text{has duality}}$$

independently of duality on $\mathcal{A}$

$$(A, B)^{\#} := (B, A)$$

$$A_0 \xrightarrow{f} A_1 \longmapsto (A_0, \# A_1)$$

$$A_0 \xrightarrow{0} A_1 \longleftarrow\!\shortmid (A_0, A_1)$$

these functors induce (using Additivity)

$$GW(\mathrm{Fun}([1], \mathcal{A})) \xrightarrow{\simeq} GW(\mathcal{H}\mathcal{A})$$

Note that a symmetric form in $\mathcal{H}\mathcal{A}$ is $(A, B) \xrightarrow{\simeq} (B, A)$,

so in fact just an object in $\mathcal{A}$, and $\perp$ corresponds to $\oplus$.

- <u>Fact</u>: $GW(\mathcal{H}\mathcal{A}) \simeq K(\mathcal{A})$ (where $K(\mathcal{A})$ is a connective spectrum)

Carefully computing all the functors one can rewrite $(*)$:

$$GW^{[n]}(\mathcal{A}) \xrightarrow{F} K(\mathcal{A}) \xrightarrow{H} GW^{[n+1]}(\mathcal{A}) \xrightarrow{\eta \cup} S^1 \wedge GW^{[n]}(\mathcal{A})$$

F - "forgetful functor"

H - "hyperbolic functor"

(algebraic Bott/Wood sequence)

We get from this sequence ${}^{\text{with } \varepsilon \text{ inserted}}$ that for $\varepsilon \in \{\pm 1\}$

$$ {}_{\varepsilon}V(\mathcal{A}) := \operatorname{hofib}\left(K(\mathcal{A}) \xrightarrow{F} {}_{\varepsilon}GW(\mathcal{A})\right) \simeq {}_{\varepsilon}GW^{[-1]}(\mathcal{A}) $$

$$ {}_{\varepsilon}U(\mathcal{A}) := \operatorname{hofib}\left(GW(\mathcal{A}) \xrightarrow{H} {}_{\varepsilon}K(\mathcal{A})\right) \simeq \Omega\, {}_{\varepsilon}GW^{[1]}(\mathcal{A}) $$

where ${}_{\varepsilon}GW(\mathcal{A})$ is $GW(\mathcal{A}, \omega, *, \varepsilon \cdot \mathrm{can})$,

so ${}_{\varepsilon}GW^{[-1]}(\mathcal{A}) \simeq {}_{-\varepsilon}GW^{[1]}(\mathcal{A})$.

- <u>Karoubi's fundamental theorem</u>: ${}_{-\varepsilon}V(\mathcal{A}) \simeq \Omega\, {}_{\varepsilon}U(\mathcal{A})$

Another application of the Bott/Wood sequence:

$$ GW_i^{[n]}(\mathcal{A}) \xrightarrow[\sim]{\eta\cup} GW_{i-1}^{[n-1]}(\mathcal{A}) \qquad i<0,\ n\in\mathbb{Z} $$

$$ K_0(\mathcal{A}) \xrightarrow{H} GW_0^{[n]}(\mathcal{A}) \xrightarrow{\eta\cup} GW_{-1}^{[n-1]}(\mathcal{A}) \simeq W^n(\mathcal{A}) \to 0 $$

(More generally, $W^{n-i}(\mathcal{A}) \cong GW_i^{[n]}(\mathcal{A})$ for $i<0$, $n\in\mathbb{Z}$.)

This gives a method to "prove" results about GW:

$$ \left.\begin{array}{c} \text{results on } W \\ + \\ \text{results on } K \end{array}\right\} \xRightarrow[\text{induction}]{\text{Karoubi}} \text{results on } GW $$

For example, if $\mathcal{A} \xrightarrow{F} \mathcal{B}$ induces $h\mathcal{A} \xrightarrow{\sim} h\mathcal{B}$ then $GW(\mathcal{A}) \xrightarrow{\sim} GW(\mathcal{B})$.

- homotopy invariance for regular schemes

  K-theory — Quillen

  Witt theory — Karoubi & Balmer

- Nisnevich descent

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

p étale

$$(Y \setminus V)_{red} \xrightarrow{\sim} (X \setminus U)_{red}$$
$$\| \qquad \|$$
$$Z$$

$\rightsquigarrow$

$$\begin{tikzcd} GW(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & GW(U) \arrow[d] \\ GW(Y) \arrow[r] & GW(V) \end{tikzcd}$$

because by Thomason

- $\mathrm{Perf}_Z(X) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is Morita exact

- $\mathrm{Perf}_Z(X) \xrightarrow{\sim} \mathrm{Perf}_Z(Y)$

- projective line bundle formula $\exists\, \beta \in GW_0^{[1]}(\mathbb{P}^1_k)$

$$GW^{[n]}(X) \oplus GW^{[n-1]}(X) \xrightarrow{\sim} GW^{[n]}(\mathbb{P}^1_X)$$
$$(x, y) \longmapsto p^*(x) + \beta \cup p^*(y)$$

this comes from "semi-orthogonal decomposition" of $\mathrm{Perf}(\mathbb{P}^1)$

$$\mathrm{Perf}(k) \xrightarrow{\pi^*} \mathrm{Perf}(\mathbb{P}^1) \longrightarrow \mathrm{Perf}(\mathbb{P}^1)/\mathrm{Perf}(k) \underset{\otimes \tilde{\beta}}{\xleftarrow{\sim}} \mathrm{Perf}(k)$$

$\tilde{\beta}$ is a particular complex with symm. form

III. Motivic representability of Grothendieck-Witt theory

Once we have
- homotopy invariance
- Nisnevich descent
- $\mathbb{P}^1$-bundle formula

we get a motivic spectrum:

E.g. $K : Sm_S \to Sp$

$$\begin{tikzcd} K(X) \arrow[r] & K(\mathbb{P}^1 \times X) \arrow[r] & K(\mathbb{P}^1 \wedge X) \\ & & K(X) \arrow[u, "\beta", "\wr"'] \end{tikzcd}$$

so $K \xrightarrow{\sim} \Omega_{\mathbb{P}^1} K$

and $KGL := (K, K, \dots)$ is a motivic spectrum

For Hermitian K-theory we have $GW^{[n]}(X)$, $n = 1,2,3,4$

and $GW^{[n]} \xrightarrow{\sim} \Omega_{\mathbb{P}^1} GW^{[n-1]}$

Or one can just define $\Omega^4_{\mathbb{P}^1}$-spectrum $KQ := (GW, GW, \dots)$

so that
$$[\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} K] \simeq K_{2q-p}(X)$$

$$[\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} KQ] \simeq GW^{[q]}_{q-p}(X)$$

The algebraic Bott sequence is rewritten as

$$\Sigma^{1,1} KQ \xrightarrow{\eta} KQ \xrightarrow{f} KGL \to \Sigma^{2,1} KQ$$

One could have defined L-theory of $\mathcal{A}$ as $\eta^{-1} GW(\mathcal{A})$

and prove general results about it as above.

We can also just define

$$KT := \operatorname{colim}(KQ \xrightarrow{\eta} \Sigma^{-1,-1} KQ \xrightarrow{\eta} \cdots)$$

represents Witt theory of schemes.

There is one more important <u>exact sequence</u>: Tate

$$KGL_{hC_2} \longrightarrow KQ \longrightarrow KT$$

where C_2-action on KGL may be equivalently described as:

- by the stable Adams operation
- explicitly by an involution on $\operatorname{colim} GL_{2n}$
- coming from $\sigma : \mathcal{H}\mathcal{A} \longrightarrow \mathcal{H}\mathcal{A}$ $\rightsquigarrow GW(\mathcal{H}\mathcal{A})$

$$(X,Y) \longmapsto (Y^*, X^*) \qquad \begin{array}{c} GW(\mathcal{H}\mathcal{A}) \\ \parallel \\ K(\mathcal{A}) \end{array}$$

Finally, to connect the motivic considerations with what we started with:

in the unstable motivic homotopy category $\mathcal{H}_\bullet(S)$

$$K_i(X) \simeq [X_+ \wedge S^i, \mathbb{Z} \times Gr_\bullet] \qquad \text{Morel-Voevodsky}$$
$$\simeq [X_+ \wedge S^i, \mathbb{Z} \times BGL]$$

$$GW_i(X) \simeq [X_+ \wedge S^i, \mathbb{Z} \times GrO_\bullet] \qquad \text{Schlichting-Trippathi}$$
$$\simeq [X_+ \wedge S^i, \mathbb{Z} \times B_{et} O]$$

where $$GrO_\bullet := \operatorname{colim}_n \underbrace{GrO_{2n}(H^n \perp H^n)}_{\substack{\text{subscheme of } Gr_{2n} \\ \text{of subbundles on which} \\ \text{the symmetric form is non-degenerate}}}$$