The homotopy limit problem away from 2

$\mathrm{char}(k) \neq 2$.

Key players: $KGL = kgl[\beta^{-1}]$ $\quad kgl = f_0 KGL \in SH^{\mathrm{eff}}_{\geq 0}$

$KQ = kq[\beta^{-1}]$ $\quad kq = \tau^{\mathrm{eff}}_{\geq 0} f_0 KQ \in SH(k)^{\mathrm{eff}}_{\geq 0}$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{f_0 \tau_{\geq 0} KQ}$

$KW = KT = KQ[\eta^{-1}]$ $\quad kw = \tau_{\geq 0} KW = kq[\eta^{-1}]$

$E \in SH(k)$: $\quad f_n E \longrightarrow E \longrightarrow f^{n-1} E$

$(*)$ $\quad \mathrm{map}(\mathbb{1}, E) \xrightarrow{\sim} \lim_n \mathrm{map}(\mathbb{1}, f^n E)$.

Example: KGL satisfies $(*)$.

<u>Convergence Thm</u> If $E \in SH(k)_{\geq c}$ and $\mathrm{vcd}_2(k) < \infty$, then $E/(2,\rho)$ satisfies $(*)$.

<u>Lemma</u> If $\mathrm{vcd}_2(k) < \infty$, $KQ/(2,\rho)$ satisfies $(*)$.

<u>Pf</u>.
$$\pi_{n,0} KW = \begin{cases} W(k) & \text{if } n \equiv 0 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

<u>Fact #1</u>: slice filtration on $\pi_{0,0} KW \equiv$ I-adic filtration on $W(k)$
where $I = \ker\left(W(k) \xrightarrow{\mathrm{rk}} \mathbb{Z}/2\right)$.

$\Longrightarrow \quad \pi_{*,0}\left(\lim_n f^n KW\right) = \pi_{*,0}(KW)^{\wedge}_I$

$\mathrm{vcd}_2(k) < n$ implies : $\quad 2^n W(k)_{\mathrm{tor}} = 0$ and $I^n \subset 2W(k)$.

$\mathrm{vcd}_2(k) < \infty \quad \Longrightarrow \quad W(k)^{\wedge}_I = W(k)^{\wedge}_2 =$ derived 2-completion

$\Longrightarrow \quad KW/2$ satisfies $(*)$

KW is η-periodic $\Longrightarrow \eta\rho = -2$ $\quad$ (from 4-th Milnor-Witt relation: $\eta\underbrace{(\eta\rho+2)}_{h = \langle 1,-1 \rangle} = 0$)

$\Longrightarrow \quad KW/(2,\rho)$ satisfies $(*)$

By convergence thm, $kq/(2,\rho)$ and $kw/(2,\rho)$ satisfy $(*)$.

$$\begin{tikzcd}
\boxed{kq} \arrow[r] \arrow[d] & KQ \arrow[r] \arrow[d] & E = \mathrm{cofib} \arrow[d] \\
\boxed{kw} \arrow[r] & \boxed{KW} \arrow[r] & \boxed{E[\eta^{-1}]}
\end{tikzcd}$$

$\square/(2,\rho)$ satisfies $(*)$.

$KQ \to KW$ is an iso on $\pi_{*,0}$ for $* < 0$

$\Rightarrow$ $E \to E[\eta^{-1}]$ is an $\pi_{*,0}$-iso for all $*$.

$\Rightarrow$ $f_0 E \xrightarrow{\sim} f_0(E[\eta^{-1}])$ ← htpy in the effective htpy t-structure

Claim: $E \in SH(k)$, E/ρ satisfies $(*)$ iff $(f_0 E)/\rho$ satisfies $(*)$.

$$\begin{tikzcd}
\mathrm{map}(\mathbb{1}, f^n(E/\rho)) & \mathrm{map}(\mathbb{1}, f^n((f_0 E)/\rho)) \arrow[l, "\sim"'] \\
\mathrm{map}(\mathbb{1}, f^n E) \arrow[u] & \mathrm{map}(\mathbb{1}, f^n f_0 E) \arrow[l, "\sim"'] \arrow[u] \\
\mathrm{map}(\mathbb{G}_m, f^{n+1} E) \arrow[u, "\rho"] & \mathrm{map}(\mathbb{G}_m, f^{n+1} f_0 E) \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$
$\square$.

Lemma (Heard)

Over any base S, $\mathbb{Z}[1/2]$ $KQ^\wedge_\eta \simeq KGL^{hC_2}$.

Pf. Wood cofiber sequence $\Sigma^{1,1} KQ \xrightarrow{\eta} KQ \xrightarrow{f} KGL$

Also, $\eta = 0$ in KGL so $KGL/\eta = KGL \oplus \Sigma^{2,1} KGL = KGL \oplus KGL$

$BC_2^{\triangleleft} \longrightarrow SH(S)$: $KQ \xrightarrow{f} KGL^{BC_2}$

mod η: $KGL \xrightarrow{\Delta} KGL \oplus KGL^{BC_2}$ limit diagram

$\Rightarrow$ $KQ \to KGL^{hC_2}$ is an η-adic equivalence.

Moreover, KGL^{hC_2} is η-complete. $\square$

Theorem If $\mathrm{vcd}_2(k) < \infty$ then

$KQ/2 \longrightarrow KGL^{hC_2}/2$

is an π_{**}-iso, i.e.,

$GW^{[n]}(k) \longrightarrow K^{[n]}(k)^{hC_2}$ ← C_2-action induced by $V \mapsto V^*[n]$ is a 2-adic equivalence.

Proof. By the lemma, it suffices to show $KQ/2 \longrightarrow KQ^{\wedge}_{\eta}/2$ is a π_{**}-iso.

Let $F = \mathrm{fib}(KQ \longrightarrow KQ^{\wedge}_{\eta})$

F is η-periodic $\Rightarrow$ $\eta \cdot \rho = -2$ on F

$\rightsquigarrow$ it suffices to show that $\mathrm{map}(\mathbb{1}, F/(2,\rho)) = 0$

$$F = \lim_{t \to \infty} \Sigma^{t,t} KQ \qquad [\eta]$$

$$= \lim_{t \to \infty} \Sigma^{-4t} KQ \qquad [\eta^4 \beta]$$

$KQ \overset{\beta}{\simeq} \Sigma^{8,4} KQ.$

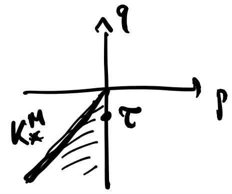
$$\mathrm{map}(\mathbb{1}, F/(2,\rho)) = \lim_t \mathrm{map}(\mathbb{1}, \Sigma^{-4t} KQ/(2,\rho)) \overset{\text{convergence lemma}}{=} \lim_n \underbrace{\lim_t \mathrm{map}(\mathbb{1}, \Sigma^{-4t} f^n(KQ)/(2,\rho))}_{A_n}$$

- $A_{<0} = 0$ because $\mathbb{1}$ is effective
- $\mathrm{fib}(A_n \to A_{n-1}) = \lim_t \mathrm{map}(\mathbb{1}, \Sigma^{-4t} s_n(KQ)/(2,\rho))$ $(*)$

Fact #2: $s_n KQ/2 = \bigoplus$ spectra of the form $\Sigma^{2n-i,n} H\mathbb{Z}/2$ with $i \geq 0$.

Now $\pi_{**} H\mathbb{Z}/2 = K^M_*(k)/2[\tau]$ $\qquad |K^M_1| = (-1,-1)$

$|\tau| = (0,-1)$



$\Rightarrow$ $\pi_{*,q} H\mathbb{Z}/(2,\rho) = 0$ if $* \geq 3$

$\overset{\text{Fact 2}}{\Longrightarrow}$ $\mathrm{map}(\mathbb{1}, s_n(KQ)/(2,\rho))$ is $(2n+2)$-truncated.

$\Rightarrow$ $\lim_{t \to \infty} (*) = 0.$ $\square$.

Corollary Let X be a qcqs $\mathbb{Z}[1/2]$-scheme of finite Krull dimension and finite punctual vcd_2. Then

$$GW^{[n]}(X) \longrightarrow K^{[n]}(X)^{hC_2}$$

is a 2-adic equivalence.

Fake Pf. Both $GW^{[n]}$ and K are Nisnevich sheaves & preserve filt. colim. of rings.

$\dim(X) < \infty \Rightarrow$ we can assume X local henselian, with closed point x.

$$K(X)/2 \xrightarrow{\sim} K(\kappa(x))/2 \qquad (\text{Gabber})$$
$$W(X) \xrightarrow{\sim} W(\kappa(x))$$
$$\left[\ \mathrm{Vect}^{\mathrm{sym}}(X) \longrightarrow \mathrm{Vect}^{\mathrm{sym}}(\kappa(x)) \quad 1\text{-connective}\ \right]$$

$\Rightarrow$ also for $GW^{[n]}/2$ by the fundamental fibre square. $\square$

<u>Slices of KQ</u>.

<u>Theorem</u> $kq/\eta = kgl$

Pf. We know $\Sigma^{1,1} KQ \xrightarrow{\eta} KQ \xrightarrow{f} KGL$

$\overset{f_0}{\rightsquigarrow}$

$$\begin{tikzcd}
\Sigma^{1,1} f_{-1} KQ \arrow[r] & f_0 KQ \arrow[r] & kgl \\
\tau^{\mathrm{eff}}_{\geq 0} \Sigma^{1,1} f_{-1} KQ \arrow[u] \arrow[r] \arrow[d, phantom, "\shortparallel"] & \tau^{\mathrm{eff}}_{\geq 0} f_0 KQ \arrow[u] \arrow[r] \arrow[d, phantom, "\shortparallel"] & kgl \arrow[u, equal] \\
\Sigma^{1,1} \tau^{\mathrm{eff}}_{\geq -1} f_{-1} KQ & kq & \pi^{\mathrm{eff}}_{0}\text{-epimorphism} \arrow[u] \\
& & GW \xrightarrow{\mathrm{rk}} \mathbb{Z}
\end{tikzcd}$$

The theorem is equivalent to:

$$kq = \tau^{\mathrm{eff}}_{\geq 0} f_0 KQ \xrightarrow{\sim} \tau^{\mathrm{eff}}_{\geq -1} f_{-1} KQ \qquad (\text{Bachmann})$$
$$\downarrow$$
$$\tilde{s}_{-1} KQ = 0$$

Philosophy: $\quad \tilde{s}_n KQ \rightsquigarrow \tau_{\geq 2n} \tau_{\leq 2n+1} KO \qquad \pi_{-2} KO = \pi_{-1} KO = 0.$ $\square$.

<u>Theorem</u> (Ananyevskiy - Röndigs - Østvær)

$$s_q kq = \begin{cases} \Sigma^{2q,q} H\mathbb{Z} \oplus \bigoplus_{\substack{2 \leq i \leq q \\ \text{even}}} \Sigma^{2q-i,q} H\mathbb{Z}/2 & q \text{ even } \geq 0 \\ \bigoplus_{\substack{1 \leq i \leq q \\ \text{odd}}} \Sigma^{2q-i,q} H\mathbb{Z}/2 & q \text{ odd } \geq 0 \\ 0 & q < 0 \end{cases}$$

Pf. s_q is exact $\Longrightarrow$ $\Sigma^{1,1} s_{q-1} kq \to s_q kq \to s_q kgl$

$q=0$: $s_0 kq = s_0 kgl = H\mathbb{Z}$

$q=1$: $\Sigma^{1,1} H\mathbb{Z} \to s_1 kq \to s_1 kgl = \Sigma^{2,1} H\mathbb{Z} \underset{n\in\mathbb{Z}}{\overset{?}{\longrightarrow}} \Sigma^{2,1} H\mathbb{Z}$

$n=2$: $KGL \to \Sigma^{2,1} KQ$ is $\Sigma^{2,1}$ of the hyperbolic map $KGL \to KQ$

$$\begin{tikzcd} K \arrow[r, "hyp"] \arrow[d, "rk"'] & GW \arrow[r, "f"] & K \arrow[d, "rk"] \\ \mathbb{Z} \arrow[rr, "2"] & & \mathbb{Z} \end{tikzcd}$$

$\Longrightarrow$ $s_1 kq = \Sigma^{1,1} H\mathbb{Z}/2$

$q=2$: $\Sigma^{2,2} H\mathbb{Z}/2 \to s_2 kq \to s_2 kgl = \Sigma^{4,2} H\mathbb{Z} \xrightarrow{0} \Sigma^{3,2} H\mathbb{Z}/2$

because $H\mathbb{Z}, H\mathbb{Z}/2 \in SH(k)^{eff\heartsuit}$

$\Longrightarrow$ $s_2 kq = \Sigma^{4,2} H\mathbb{Z} \oplus \Sigma^{2,2} H\mathbb{Z}/2$. $\square$

$KQ = kq[\beta^{-1}]$, $KW = KQ[\eta^{-1}]$.

<u>Corollary</u>

$$s_q KQ = \begin{cases} \Sigma^{2q,q} H\mathbb{Z} \oplus \bigoplus_{i\geq 2 \text{ even}} \Sigma^{2q-i,q} H\mathbb{Z}/2 & q \text{ even} \\ \bigoplus_{i\geq 1 \text{ odd}} \Sigma^{2q-i,q} H\mathbb{Z}/2 & q \text{ odd} \end{cases}$$

$$s_q KW = \bigoplus_{i\equiv q \bmod 2} \Sigma^{2q-i,q} H\mathbb{Z}/2 .$$