

# Localization & Devissage in GW.

Classical story.

$F$  field, in any char

Lemma (Witt)  $W(F) =$  free ab gp on  $\langle \alpha \rangle$

$\alpha \in F^\times$  subject to 3 relations:

1)  $\langle \alpha \rangle = \langle \alpha p^2 \rangle \quad p \neq 0 \quad p \in F.$

2)  $\langle \alpha \rangle + \langle -\alpha \rangle = 0.$

3)  $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha p(\alpha + \beta) \rangle + \langle \alpha + \beta \rangle.$

Given this presentation, we can construct a "boundary map"

of the form

$$W(F) \xrightarrow{\partial_\pi} W(k)$$

When

$F$  has a discrimin val

$$v: F^\times \rightarrow \mathbb{Z},$$

$$\pi \in R \subseteq F$$

$$k = R/\mathfrak{m}.$$

*this dependence is important*

Lemma (Springer, Kacbuch.)  $\exists$  map. (addition homom.)

$$\partial_\pi: W(F) \longrightarrow W(k).$$

such that.

$$\partial_\pi(\langle \pi^i u \rangle) = \begin{cases} \langle \bar{u} \rangle & i \equiv 1 \pmod{2} \\ 0 & i \equiv 0 \pmod{2}. \end{cases}$$

the proof of this goes by verifying gen's + rel's.

This is very useful.

Lemma  $I(F) := \ker(W(F) \rightarrow \mathbb{Z}/2).$

$$I(\mathbb{F}_q) = \begin{cases} 0 & q \text{ even} \\ \mathbb{Z}/2. & q \text{ odd.} \end{cases}$$

$$W(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2 & q \text{ even.} \\ \mathbb{Z}/4 & q \equiv 3 \pmod{4} \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & q \equiv 1 \pmod{4}. \end{cases}$$

"Pf" .  $I(\mathbb{F}_q) \cong \mathbb{F}_q^\times / 2(\mathbb{F}_q^\times)$

·  $q \equiv 3 \pmod{4}$

then  $-1$  is not square

$\langle 1 \rangle \oplus \langle -1 \rangle \neq 0 \Rightarrow \langle 1 \rangle$  <sup>not</sup> order 2.

· on the other hand if

$q \equiv 1 \pmod{4}$

then  $-1$  is a sq

so  $\langle -1 \rangle = \langle 1 \rangle$

$\Rightarrow \langle 1 \rangle \oplus \langle 1 \rangle = 0.$

□

$\leadsto$  gives a completion of  $W(\mathbb{Q})$ .  
split.

Thm. There is an exact seq:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & W(\mathbb{Z}) & \longrightarrow & W(\mathbb{Q}) & \xrightarrow{\partial_\pi} & \bigoplus W(\mathbb{F}_p) \longrightarrow 0 \\
 & & \searrow \scriptstyle 2 & & \nearrow \scriptstyle \text{can} & & \\
 & & \mathbb{Z} & & & & 
 \end{array}$$

In fact we have versions of this for  $D = \text{adequate domain}$

$$0 \rightarrow W(D) \rightarrow W(F) \rightarrow \bigoplus W(D/p) \quad (\rightarrow \ell(D)_k)$$

The goal of this talk is to explain these ex seq. in terms of the theory of Spectra.

How to do this intuitively:

$$\mathbb{Z} \hookrightarrow X \quad \text{Smooth / } S$$

then:  $X /_{X \setminus \mathbb{Z}} \simeq \text{Th}(v_{\mathbb{Z}})$

If  $E = \text{csh. thg. thm}$

$$E_{\mathbb{Z}}(X) = [X /_{X \setminus \mathbb{Z}}, E]$$

$\leadsto$  get a white seq.

$$E_z(x) \rightarrow E(x) \rightarrow E(X|z)$$

$$\int_2 \rightsquigarrow E \text{ ordering ths. } j \text{ o/w}$$

$$E^{-n}(z) \quad \text{we still get a twisted form of } E^{-n}(z)$$

• this is what this Devissage thm is all about.

Thm (Quillen's Devissage.)  $i: A \subseteq B$  inclusion

of ab. cats such that.

1)  $A$  is an exact ab. subcat,  
closed under sub's and quot's

2) any  $B \in \mathcal{B}$  has a finite

filtration:

$$0 = B_r \subsetneq B_{r-1} \subsetneq \dots \subsetneq B_1 \subseteq B_0 = B.$$

$$\text{such } B_j / B_{j-1} \in A.$$

$$\text{the } K(A) \stackrel{i}{\simeq} K(B).$$

Example  $\mathcal{A} = \text{Coh}(X)$ .

$$\begin{aligned}\text{Coh}(X \text{ on } Z) &= \{ \mathcal{I}_Z^n \text{-tors. obj} \} \\ &= \ker(\text{Coh}(X) \rightarrow \text{Coh}(X \setminus Z)).\end{aligned}$$

then Devissage says that

$$K(\text{Coh}(X \text{ on } Z)) = K(\text{Coh}(Z))$$

Therefore if  $Z, X$  regular then get ex seq.

$$K(Z) \rightarrow K(X) \rightarrow K(X \setminus Z)$$

A notable consequence:

$$\begin{aligned} \bigoplus_{p \text{ prime}} K(\mathbb{F}_p) &\rightarrow K(\mathbb{Z}) \rightarrow K(\mathbb{Q}) \\ &\rightarrow \bigoplus_{p \text{ prime}} K(\mathbb{F}_p)[\mathbb{Z}]. \end{aligned}$$

$R =$  Dedekind ring.

Input:  $M =$  line bundle on  $\text{Spec } R$

w/ involution

Thm (#9) We have cofiber sequences:

$$1) \bigoplus_{\substack{p \text{ prime} \\ \text{prime of} \\ R}} CW(\mathbb{F}_p, I_{p|M}^s)^{[m-1]} \rightarrow CW(R, (I_M^s)^{[m]}) \\ \rightarrow CW(F, (I_{M_F}^s)^{[m]})$$

2) as above w/  $L(-, I^s)$

Remark / Construction. The main result is really a  
Devissage thm for  $CW/L$ .

$R =$  Dedekind ring, for each  $p \in \text{Spa } R \setminus \{0\}$ .

Choose a uniformizer  $\pi_p$  of  $R_p$ , w

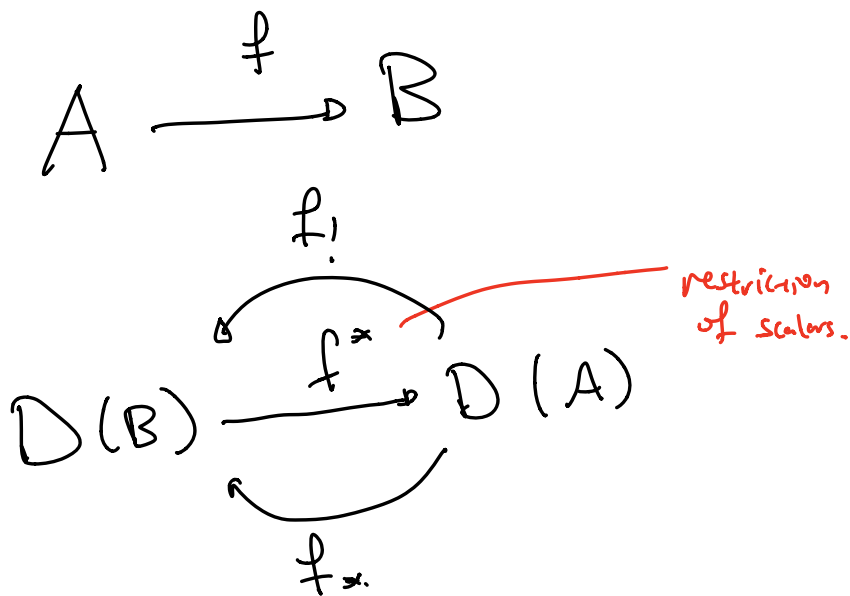
Construct a map:

$$\gamma : \bigoplus_{p \in \text{Spa } R \setminus \{0\}} (D^p(\mathbb{F}_p), (I_{p|M}^s)^{[m-1]}) \\ \rightarrow (D^p(R)_{\text{tors}}, (I_M^s)^{[m]})$$

$$\ker(D^p(R) \rightarrow D^p(F))$$

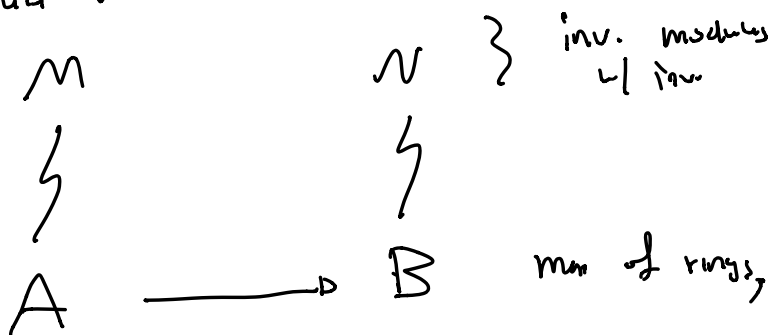
the main thm asserts that the induced map on  $GW/L$  is an  $\cong$ .

We construct this map. :



$$f_1 = \otimes_A B \quad f_x = \text{Hom}_A(B, -)$$

to construct the abelian functor.





assume that  $B$  is perfect so that

$f^*$  restricts to a functor on

$$D^p(B) \xrightarrow{f^*} D^p(A).$$

$$\begin{array}{ccc} \text{OS}_{\perp N}(X) = \text{hom}_{B \otimes B} (X \otimes X, N)^{h\mathbb{Z}_2} & & (f^* \otimes f^*)(N) \\ \downarrow & \nearrow & \downarrow \star \\ \text{OS}_{\perp M}(f^* X) = \text{hom}_{A \otimes A} (f^* X \otimes f^* X, M)^{h\mathbb{Z}_2} & & M. \end{array}$$

From this  $(\star)$  data we get a map.

$$f^* N \longrightarrow M$$

$$\Leftrightarrow N \longrightarrow f_* M.$$

Lemma.  $N \longrightarrow f_* M$  is an equiv. iff.

$f^*$  is Poincaré i.e.,

the map

$$f^* D \longrightarrow D f^* \text{ is invertible.}$$

We going to apply this to the map.

$$R \xrightarrow{p} R/p = \mathbb{F}_p.$$

$$\leadsto D^p(\mathbb{F}_p) \xrightarrow{p^*} D^p(R)$$

We want to make a map.

$$(D^p(\mathbb{F}_p), ?) \longrightarrow (D^p(R), (I_M^p)^{\text{inv}})$$

Lemma.  $R$  Dedekind,  $p \subseteq R$  prime.

$$p: R \longrightarrow R/p.$$

then we have an equivalence:

$$(p_x R)[1] \simeq p_x(p^{-1})$$

$$\simeq \mathbb{F}_p$$

inv. module in  $R$   
b/c  $R$  Dedekind

$$p \neq \emptyset \quad p \longrightarrow R \longrightarrow \mathbb{F}_p$$

$$\begin{array}{c} \hookrightarrow \text{Hom}_R(-, R) \\ P^* P_* R \longrightarrow R \xrightarrow{\pi^{-1}} \end{array} \quad \begin{array}{c} \text{cyclic seq.} \\ \downarrow \\ (P^* P_* R)[1] \end{array}$$

$$\begin{aligned} \Rightarrow (P^* P_* R)[1] &\simeq \mathbb{F}_p \otimes_R R^{-1} \\ &\simeq P^* P_*(\pi^{-1}) \end{aligned}$$

$$\begin{array}{ccc} P & \rightarrow & R & \rightarrow & \mathbb{F}_p \\ & & \otimes & & \\ & & \pi^{-1} & & \end{array}$$

On the other hand,  $P^*$  is fully faithful on discrete modules  $\Rightarrow P_* R[1] \simeq P_*(\pi^{-1})$ .

Now, we know abstractly that

$$P_*(R)[1] \simeq \mathbb{F}_p$$

To explicit the dependence on  $\pi$ , we apply  $P^*$

$$P^* P_*(R)[1] \simeq \text{Hom}_R(\mathbb{F}_p, R)[1]$$

$$\simeq \text{Cof}(R \xrightarrow{\pi} R)$$

$$= R/\pi \simeq \mathbb{F}_p$$

$\square$

From the above, we see that

we have equivalences:

$$\begin{array}{ccc}
 P_! M \simeq \mathbb{F}_p \otimes_{\mathbb{R}} M & \xrightarrow{\pi} & P_! \mathbb{R} \langle \mathbb{D} \rangle \otimes_{\mathbb{R}} M \\
 \downarrow & & \downarrow \\
 & \xrightarrow{\sim} & P_! M \langle \mathbb{D} \rangle.
 \end{array}$$

equiv. dep. on  $\pi$ .

so adjoint this equiv to get a map.

$$P^* P_! M \longrightarrow M \langle \mathbb{D} \rangle.$$

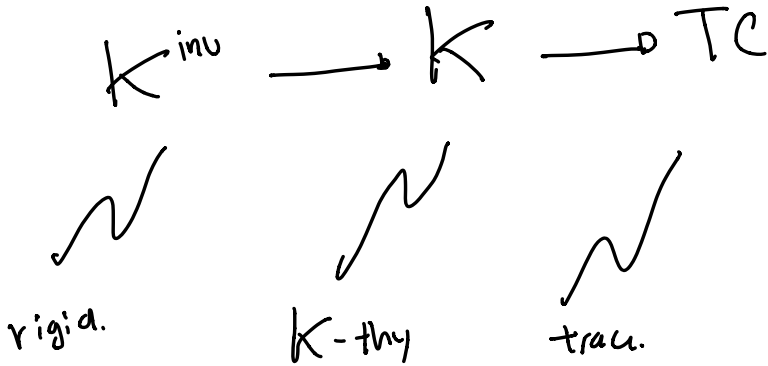
this is the data needed to define a Poisson structure

$$(D^p(\mathbb{F}_p), \mathcal{I}_{P_! M}^S) \longrightarrow (D^p(\mathbb{R}), \mathcal{I}_{M \langle \mathbb{D} \rangle}^S).$$

$$\begin{array}{ccc}
 & \nearrow & \mathcal{I}_M^S \langle \mathbb{D} \rangle \\
 & & \parallel \\
 & & \mathcal{I}_M^S \langle \mathbb{D} \rangle \\
 & \searrow & \\
 & & (D^p(\mathbb{R})_{p\text{-tors}}, (\mathcal{I}_M^S)^{\langle \mathbb{D} \rangle})
 \end{array}$$

this is the dev. map.

Remark



Expect • purity / divisibility for  $K$ -type invariants

• not expected for trac type.

• not for rigid inv either.

• Instead of asking for divisibility for rigid type inv. we have rigidity.

•  $W^2$  rigid type inv.

• Kato  $I^j W^2 / I^{j+1} W^2 \cong \tilde{V}^j$   
 $= \text{coker} (\Omega^j \xrightarrow{1-C^{-1}} \Omega^j/a)$   
filters  $K^{inv}$ .

Prop 2.13 (#9)  $R$   $I$ -adically complete ring  
 with

$$\text{thm } L^1(R) \simeq L^1(R/I).$$

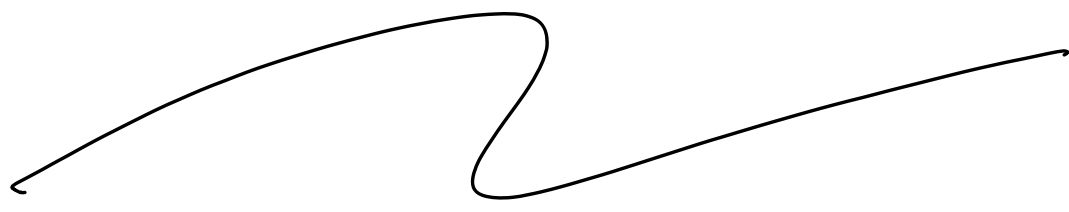
On the  $L^1$  cannot have divisibility, if it did then

$$\Omega L^1(\mathbb{F}_2) \rightarrow L^1(\mathbb{Z}) \rightarrow L^1(\mathbb{Z}[\frac{1}{2}])$$

$$\rightarrow \pi_3 L^1(\mathbb{F}_2) \rightarrow L_2^1 \mathbb{Z} \rightarrow L_2^1(\mathbb{Z}[\frac{1}{2}])$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad 0 \quad \quad \quad 0$$



Pf of #9 - thm.

I.) Recall last semester.

$$K(\mathcal{L}, \mathcal{I})_{h\mathcal{L}_2} \rightarrow \text{GW}(\mathcal{L}, \mathcal{I}) \rightarrow L(\mathcal{L}, \mathcal{I}).$$

is a fiber seq. from the  $\square$ :

$$\begin{array}{ccc} \text{GW}(\mathcal{L}, \mathcal{I}) & \longrightarrow & L(\mathcal{L}, \mathcal{I}) \\ \text{forg} \downarrow & & \downarrow \cong \\ K(\mathcal{L}, \mathcal{I})^{h\mathcal{L}_2} & \longrightarrow & K(\mathcal{L}, \mathcal{I})^{t\mathcal{L}_2}. \end{array}$$

From this we only need to prove for  $K, L$  separately.

II) For  $K$ : since  $K, \text{GW}, L$

are finitary. we can prove a version where we consider only finitely many primes in  $\mathbb{R}$ .

In this case we may write

$$\begin{aligned} & \bigoplus D^p(\mathbb{F}_p) \\ & \simeq \prod D^p(\mathbb{F}_p) \end{aligned}$$

✓

embed this  $\oplus$  - term w/ product t-str.

Such that

$$\mathcal{H} \simeq \prod \text{Vect}_{\mathbb{F}_p}$$

$$\simeq \text{Mod}_{\prod \mathbb{F}_p}^{\text{f.g.}}$$

On the other hand  $\sim$ ,  $R, R_S$  both have global dim  $\leq 1$ . So the standard t-str.  $\simeq \begin{matrix} D(R), \end{matrix}$   
 $D(R_S)$

restricts on  $D^p(R), D^p(R_S)$ .

and the map  ~~$D^p(R)$~~   $R \rightarrow R_S$  flat

$$\Rightarrow D^p(R) \rightarrow D^p(R_S)$$

Commutative w/ conn, tr. objects.

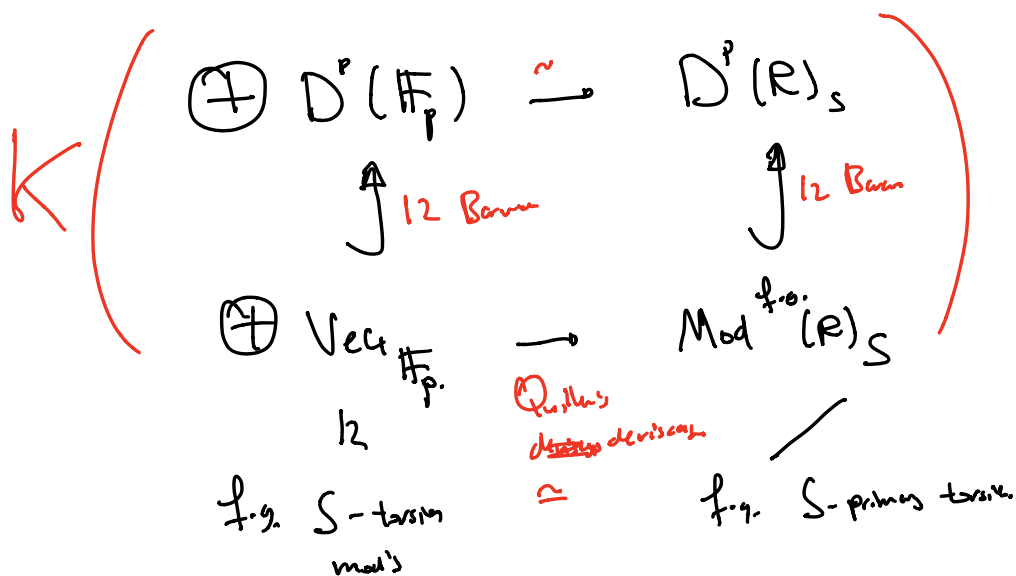
$\Rightarrow D^p(R)_S$  has a t-structure such that

$$(\ker : D^p(R) \rightarrow D^p(R_S))$$

$$D^p(R)_S \hookrightarrow D^p(R) \text{ conn w/ } \tau_{\leq 0}, \tau_{\geq 0}$$



$\Rightarrow$   $\heartsuit$  of this t-structure  $D^p(R)_S$   
 $\simeq$   $S$ -primary torsion modules in  $\text{Mod}(R)$ .



III. Onto symmetric  $L$ -thy.

$$L(\mathcal{L}, \mathcal{I}^{(1)}) \simeq \sum L(\mathcal{L}, \mathcal{I}).$$

Assume that  $m=0$ .

$$D_{P, M} = \text{Hom}(-, M \otimes_{\mathbb{F}_p}). \qquad P: R \rightarrow \mathbb{F}_p.$$

$$D_{P, M}: D^p(\mathbb{F}_p) \rightarrow D^0(\mathbb{F}_p)^{\text{op.}}$$

this surps  $\mathcal{O}$ -conn. objects  $\rightarrow$   
 $\mathcal{O}$ -tr objects

$$D_{p!M}(\geq 0) \subseteq \leq 0.$$

bc  $\mathbb{F}_p$  has global dim 0.

Now  $\sum D_M$  on  $D^p(R)$

$$\text{Std } \geq 0 \rightarrow \leq 1$$

by glob. dim of  $R$ .

However, she  $M$  tors free:

$$\pi_1 \sum D_M(X) = \pi_0 \text{hom}_R(X, M[0])$$

$$= [X_{\leq 0}, M[0]]$$

$$= \text{Hom}(\underbrace{\pi_0(X)}_{S\text{-torsion}}, \underbrace{M}_{\text{tors. free}})$$

$$= 0.$$

In other words if  $w \rightarrow \sum D_M$  on  $D(R)_S$

then we do surps  $\geq 0 \rightarrow \leq 0$ .

In diagram:

$$\begin{array}{ccc} (D^p(R))_{\geq 0} & \longrightarrow & (D^p(R))_{\leq 0} \\ \downarrow & & \downarrow \\ (D^p(R))_{\geq 0} & \xrightarrow{D_m} & D^p(R)_{\leq 1} \end{array}$$

"duality does on better on torsion obs"

Summary  $\mathcal{L}$  a stable  $\infty$ -cat.

$$D: \mathcal{L}^{\text{op}} \rightarrow (\mathcal{L}^{\text{op}})^{\text{op}}$$

Set  $\mathcal{D}_D^{\text{gs}}$

$$\mathcal{I}_{\mathcal{D}^{\text{op}}}^{\text{gs}}: X \mapsto \text{hom}_{\mathcal{L}^{\text{op}}}(X, \mathcal{D}^{\text{op}} X)$$

$$\simeq W(\mathcal{L}^{\text{op}}, \mathcal{I}_{\mathcal{D}^{\text{op}}}^{\text{gs}})$$

From last time, in this situation,

$$L_n(\mathcal{L}, \mathcal{I}_D^{\text{gs}}) = \begin{cases} W(\mathcal{L}^{\text{op}}, \mathcal{I}_{\mathcal{D}^{\text{op}}}^{\text{gs}}) & n \equiv 0 \pmod{4} \\ W(\mathcal{L}, \mathcal{I}_{-D^{\text{op}}}^{\text{gs}}) & n \equiv 2 \pmod{4} \end{cases}$$

~~Goal~~ Suffices to prove denseness for Witt gr's! <sup>0</sup> <sup>odd.</sup>

$$\left. \begin{aligned} W(\text{Vect}(\mathbb{F}_p), \mathcal{I}_n) \\ \simeq W(\mathcal{I}_r\text{-tors}, \mathcal{I}_{\Sigma n}) \end{aligned} \right\} \text{classical}$$

$\square$ .