Localization & Devissage in GW.

Classical story.

F field, in any char

Lemma (Witt) $W^s(F) =$ free ab gp on $\langle\alpha\rangle$

$\alpha \in F^\times$ subject to 3 relations:

1) $\langle\alpha\rangle = \langle\alpha\beta^2\rangle \quad \beta \neq 0 \quad \beta \in F.$

2) $\langle\alpha\rangle + \langle-\alpha\rangle = 0.$

3) $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha\beta(\alpha+\beta)\rangle. + \langle\alpha+\beta\rangle.$

Given this presentation, we can construct a "boundary map"

of the form

$$W(F) \xrightarrow{\partial_\pi} W(k)$$

this dependence is important

When
- F has a discrete val

  $v: F^\times \longrightarrow \mathbb{Z}$,

  $\pi \in R \subseteq F$

- $k = R/m.$

<u>Lemma</u> (Springer, Knebusch.) $\exists$ map. (additive homom.)

$$\partial_\pi : W(F) \longrightarrow W(k).$$

such that.

$$\partial_\pi(\langle \pi^i u \rangle) = \begin{cases} \langle \bar{u} \rangle & i \equiv 1 \mod 2 \\ 0 & i \equiv 0 \mod 2. \end{cases}$$

the proof of this goes by verifying gen's + rel's.

This is very useful.

<u>Lemma</u> $I(F) := \ker (W(F) \to \mathbb{Z}/2)$.

$$I(\mathbb{F}_q) = \begin{cases} 0 & q \text{ even} \\ \mathbb{Z}/2. & q \text{ odd.} \end{cases}$$

$$W(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2 & q \text{ even.} \\ \mathbb{Z}/4 & q \equiv 3 \mod 4 \\ \mathbb{Z}/2 \times \mathbb{Z}/2 & q \equiv 1 \mod 4. \end{cases}$$

"Pf". $I(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{F}_q^\times / 2(\mathbb{F}_q^\times)$

- $q \equiv 3 \bmod 4$

  then -1 is not square

  $\langle 1 \rangle \oplus \langle 1 \rangle \neq 0 \Rightarrow \langle 1 \rangle$ not order 2.

- On the other hand if

  $q \equiv 1 \bmod 4$

  then -1 is a sq

  so $\langle -1 \rangle = \langle 1 \rangle$

  $\Rightarrow \langle 1 \rangle \oplus \langle 1 \rangle = 0.$ $\boxtimes$

$\rightsquigarrow$ gives a computation of $W(\mathbb{Q})$.

Thm. There is an split exact seq:

$$\begin{tikzcd}
0 \arrow[r] & W(\mathbb{Z}) \arrow[r] & W(\mathbb{Q}) \arrow[r, "\partial_\pi"] & \bigoplus W(\mathbb{F}_p) \arrow[r] & 0 \\
 & \mathbb{Z} \arrow[u, "\sim"] \arrow[ur, "\text{can}"'] & & &
\end{tikzcd}$$

In fact we have versions of this for D = dedekind domain.

$$0 \to W(D) \to W(F) \to \bigoplus W(D/p)$$

$(\to Cl(D)/2)$.

the goal of this talk is to explain these ex seq. ; refine them to lvl of spectra.

How to do this intuitively:

$Z \hookrightarrow X$ smooth $/ S$

then: $X/_{X \setminus Z} \simeq Th(\nu_Z)$.

If E = csh. thy then

$$E_Z(X) = [X/_{X \setminus Z}, E]$$

$\leadsto$ get a cofib seq.

$$E_Z(X) \longrightarrow E(X) \longrightarrow E(X \setminus Z)$$

$\Big|\wr$ 2 ~ of E orienting thy. ; o.k.

$E^{-n}(Z)$ we still get a twisted form of $E^{-n}(Z)$

- this is what this Devissage thm is all about.

<u>Thm</u> (Quillen's Devissage.) $i : \mathcal{A} \subseteq \mathcal{B}$ inclusion of ab. cat's such that.

1) $\mathcal{A}$ is an exact ab subcat, closed under sub's and quot's

2) any $B \in \mathcal{B}$ has a finite filtration:

$$0 = B_r \subsetneq B_{r-1} \subsetneq \cdots \subsetneq B_1 \subseteq B_0 = B.$$

such $B_j / B_{j-1} \in \mathcal{A}$.

then $K(\mathcal{A}) \overset{i}{\simeq} K(\mathcal{B})$.

Example $\mathcal{A} = \mathrm{Coh}(X)$.

$$\mathrm{Coh}(X \text{ on } Z) = \{ \mathcal{I}_Z^n\text{-tors. obj} \}$$

$$= \ker\left(\mathrm{Coh}(X) \to \mathrm{Coh}(X \setminus Z) \right).$$

then Devissage says that

$$K(\mathrm{Coh}(X \text{ on } Z)) \simeq K(\mathrm{Coh}(Z))$$

Therefore if Z, X regular then get ex seq.

$$K(Z) \to K(X) \to K(X \setminus Z)$$

A notable consequence:

$$\bigoplus_{p \text{ primes}} K(\mathbb{F}_p) \to K(\mathbb{Z}) \to K(\mathbb{Q})$$

$$\to \bigoplus_{p \text{ prime}} K(\mathbb{F}_p)[1].$$

R = Dedekind ring.

Input: M = line bundle on $\mathrm{Spec}\, R$

w/ involution

<u>Thm</u> (#9) We have cofiber sequences:

1) $\bigoplus_{\substack{p \text{ nonzero} \\ \text{primes of} \\ R}} GW(\mathbb{F}_p, \mathcal{I}^{s}_{p_! M}{}^{[m-1]}) \longrightarrow GW(R, (\mathcal{I}^s_M)^{[m]})$

$\longrightarrow GW(F, (\mathcal{I}^s_{M_F})^{[m]})$.

2) as above w/ $L.(-, \mathcal{I}^s)$

<u>Remark / Construction</u>. The main result is really a Devissage thm for GW / L.

R = Dedekind ring, for each $p \in \mathrm{Spec}\, R \setminus \{0\}$. Choose a uniformizer π_p of R_p, we construct a map:

$\psi : \bigoplus_{p \in \mathrm{Spec}\, R \setminus \{0\}} (D^p(\mathbb{F}_p), (\mathcal{I}^s_{p_! M})^{[m-1]})$

$\longrightarrow (D^p(R)_{\mathrm{tors}}, (\mathcal{I}^s)^{[m]}_M)$.

$$\ker\left(D^p(R) \to D^p(F)\right)$$

the main thm asserts that the induced map on GW/L is an $\simeq$.

We construct this map:

$$A \xrightarrow{f} B$$

$$\begin{tikzcd} D(B) \arrow[r, "f^*"] & D(A) \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_*"] \end{tikzcd}$$

restriction of scalars.

$$f_! = \otimes_A B \qquad f_* = \mathrm{Hom}_A(B,-)$$

to construct the above functor

$$\begin{tikzcd} M \arrow[d, squiggly, no head] & N \arrow[d, squiggly, no head] \\ A \arrow[r] & B \end{tikzcd}$$

} inv. modules w/ inv.

map of rings,

assume that B is perfect so that

f^* restricts to a functor on

$$D^p(B) \xrightarrow{f^*} D^p(A).$$

$$\begin{tikzcd}
I_N^{\varphi s}(X) = \hom_{B\otimes B}(X\otimes X, N)^{hC_2} \arrow[d] & (f^*\otimes f^*)(N) \arrow[d, "\star"] \\
I_M^{\varphi s}(f^*X) = \hom_{A\otimes A}(f^*X\otimes f^*X, M)^{hC_2} & M.
\end{tikzcd}$$

From this $(\star)$ data we get a map.

$$f^*N \longrightarrow M$$

$$\Longleftrightarrow \cancel{f} \; N \longrightarrow f_*M.$$

Lemma. $N \longrightarrow f_*M$ is an equiv iff.

f^* is Poincare i.e.,

the map

$$f^*D \longrightarrow Df^*$$ is invertible.

We going to apply this to the map.

$$R \xrightarrow{p} R/\mathfrak{p} = \mathbb{F}_p.$$

$$\rightsquigarrow \quad D^p(\mathbb{F}_p) \xrightarrow{p^*} D^p(R)$$

we want to make a map.

$$(D^p(\mathbb{F}_p), ?) \longrightarrow (D^p(R), (\mathbb{1}^{\otimes}_{M})^{[1]}).$$

Lemma. R Dedekind, $\mathfrak{p} \subseteq R$ prime.

$$p : R \longrightarrow R/\mathfrak{p}.$$

then we have an equivalence:

$$(p_* R)[1] \simeq p_!(\mathfrak{p}^{-1})$$

inv. module in R b/c R Dedekind

$$\overset{\pi_p}{\simeq} \mathbb{F}_{\mathfrak{p}}$$

Pf) $\mathfrak{p} \longrightarrow R \longrightarrow \mathbb{F}_{\mathfrak{p}}$

$$\wr \ \hom_R(-, R).$$

$$p^* p_* R \longrightarrow R \longrightarrow \mathfrak{p}^{-1} \qquad \text{cofiber seq.}$$

$$\mathfrak{p}^{-1} \longrightarrow (p^* p_* R)[1]$$

$$\Longrightarrow \quad (p^* p_* R)[1] \simeq \mathbb{F}_p \otimes_R \mathfrak{p}^{-1}$$

$$\simeq p^* p_! (\mathfrak{p}^{-1}).$$

$$\mathfrak{p} \to R \to \mathbb{F}_p$$
$$\otimes$$
$$\mathfrak{p}^{-1}$$

On the other hand, p^* is fully faithful on discrete modules $\Longrightarrow$ $p_* R[1] \simeq p_! (\mathfrak{p}^{-1})$.

Now, we know abstractly that

$$p_*(R)[1] \simeq \mathbb{F}_p.$$

To explicate the dependence on π, we apply p^*

$$p^* p_* (R)[1] \simeq \mathrm{Hom}_R(\mathbb{F}_p, R)[1]$$

$$\simeq \mathrm{cof}\left(R \xrightarrow{\pi} R\right)$$

$$\simeq R/\pi \simeq \mathbb{F}_p.$$

$\blacksquare$

From the above, we see that

we have equivalences:

$$\begin{tikzcd} p_! M \simeq \mathbb{F}_p \otimes_R M \arrow[r, "\pi", "\sim"'] \arrow[dr, bend right, "\sim"'] & p_* R[1] \otimes_R M \arrow[d, "\sim"] \\ & p_* M[1]. \end{tikzcd}$$

equiv dep. on π.

⇝ adjoint this equiv to get a map

$$p^* p_! M \longrightarrow M[1].$$

this is the data needed to define a Poincaré functor

$$\begin{tikzcd} \left(D^b(\mathbb{F}_p), Ϙ^s_{p_! M}\right) \arrow[r] \arrow[dr, bend right] & \left(D^p(R), Ϙ^s_{M[1]}\right). \\ & \left(D^p(R)_{p\text{-tors}}, \left(Ϙ^s_M\right)^{[1]}\right) \arrow[u] \end{tikzcd}$$

$$Ϙ^s_{M[1]} = \left(Ϙ^s_M\right)^{[1]}$$

this is the dev. map.

Remark

$$K^{inv} \longrightarrow K \longrightarrow TC$$

rigid. K-thy trace.

<u>Expect</u>

- purity / devissage for K-type invariants
- not expected for trace type.
- not for rigid inv either.

- Instead of asking for devissage for rigid type inv. we have rigidity.
  - W^2 rigid type inv.
  - <u>Kato</u> $I^j W^2 / I^{j+1} W^2 \simeq \tilde{V}^j = \mathrm{coker}\left(\Omega^j \xrightarrow{1-C^{-1}} \Omega^j/a\right)$

filters $K^{inv.}$

Prop 2.1.13 (#9, Wall) R I-adically complete ring

then $L^q(R) \simeq L^q(R/I)$.

On the L^q cannot have dévissage, if it did then

$$\Omega L^q(\mathbb{F}_2) \longrightarrow L^q(\mathbb{Z}) \longrightarrow L^q(\mathbb{Z}[\tfrac{1}{2}])$$

$$\rightsquigarrow \quad \underset{\substack{\parallel \\ 0}}{\pi_3 L^q(\mathbb{F}_2)} \longrightarrow L^q_2 \mathbb{Z} \longrightarrow \underset{\substack{\parallel \\ 0}}{L^q_2(\mathbb{Z}[\tfrac{1}{2}])}$$

Pf of #9-thm).

I.) Recall last semester.

$$K(\mathcal{C},\mathfrak{P})_{hC_2} \to GW(\mathcal{C},\mathfrak{P}) \to L(\mathcal{C},\mathfrak{P}).$$

is a fiber seq. from the $\square$:

$$\begin{tikzcd} GW(\mathcal{C},\mathfrak{P}) \arrow[r] \arrow[d, "\text{forg}"'] & L(\mathcal{C},\mathfrak{P}) \arrow[d, "\Xi"] \\ K(\mathcal{C},\mathfrak{P})^{hC_2} \arrow[r] & K(\mathcal{C},\mathfrak{P})^{tC_2}. \end{tikzcd}$$

From this we only need to prove for K, L separately.

$\underline{\text{II}}$) For K: since K, GW, L are finitary. we can prove a version where we consider only finitely many primes in R.

In this case we may write

$$\bigoplus D^p(\mathbb{F}_p)$$

$$\simeq \prod D^p(\mathbb{F}_p)$$

$$\sim$$

endow this $\bigoplus$ - term w/ product t-str.

such that

$$\heartsuit \simeq \prod \mathrm{Vect}_{\mathbb{F}_p}$$

$$\simeq \mathrm{Mod}^{f.g.}_{\prod \mathbb{F}_p}.$$

On the other hand, R, R_S both have global dim ≤ 1. So the standard t-str. on $D(R)$, $\cancel{R_S}$, $D(R_S)$

restricts on $D^p(R)$, $D^p(R_S)$.

and the map $\cancel{D^p(R)}$ $R \to R_S$ flat

$$\Rightarrow D^p(R) \to D^p(R_S)$$

commutes w/ conn, tr. objects.

$\Rightarrow D^p(R)_S$ has a t-structure such that

$$\left(\ker : D^p(R) \to D^p(R_S) \right)$$

$D^p(R)_S \hookrightarrow D^p(R)$ comm w/ $\tau_{\leq 0}, \tau_{\geq 0}$.

$\Rightarrow$ $\heartsuit$ of this t-structure $D^p(R)_S$

$\simeq$ S-primary torsion modules in $\mathrm{Mod}(R)$.

$$K\left(\begin{tikzcd} \bigoplus D^p(\mathbb{F}_p) \arrow[r, "\sim"] & D^p(R)_S \\ \bigoplus \mathrm{Vect}_{\mathbb{F}_p} \arrow[r] \arrow[u, hook, "\text{12 Barr}"] & \mathrm{Mod}^{f.g.}(R)_S \arrow[u, hook, "\text{12 Barr}"] \end{tikzcd}\right)$$

$\wr\!\wr$ — f.g. S-torsion mod's

Quillen dévissage $\simeq$ f.g. S-primary torsion.

III. Onto symmetric L-thy.

$$L(\mathcal{C}, \mathfrak{P}^{[1]}) \simeq \Sigma L(\mathcal{C}, \mathfrak{P}).$$

Assume that $m=0$.

$$D_{p^! M} = \mathrm{Hom}(-, M \otimes \mathbb{F}_p). \qquad p: R \to \mathbb{F}_p.$$

$$D_{p^! M} : D^p(\mathbb{F}_p) \to D^p(\mathbb{F}_p)^{op.}$$

this swaps 0-conn. objects to

0-tr objects

$$D_{p!M}(\geq 0) \subseteq \leq 0.$$

b/c $\mathbb{F}_p$ has global dim 0.

Now ΣD_M on $D^p(R)$

send $\geq 0 \rightsquigarrow \leq 1$

by glob. dim of R.

However, since M tors free:

$$\pi_1 \Sigma D_M(X) = \pi_0 \hom_R(X, M[0])$$

$$= [X_{\leq 0}, M[0]]$$

$$= \mathrm{Hom}(\underbrace{\pi_0(X)}_{S\text{-torsion}}, \underbrace{M}_{\text{tors. free}})$$

$$= 0.$$

In other words if we res. ΣD_M on $D(R)_S$

then we do swap $\geq 0 \rightsquigarrow \leq 0$.

In diagram:

$$\begin{tikzcd} (D^p(R)_S)_{\geq 0} \arrow[r] \arrow[d] & (D^p(R))_{\leq 0} \arrow[d] \\ (D^p(R))_{\geq 0} \arrow[r, "D_M"'] & D^p(R)_{\leq 1} \end{tikzcd}$$

" duality does om better on torsion obs"

Summary: $\mathcal{C}$ a stable ∞-cat.

$$D : \mathcal{C}^{\heartsuit} \to (\mathcal{C}^{\heartsuit})^{op}$$

Set Ϙ^{gs}_{D}

$$\text{Ϙ}^{gs}_{D^{\heartsuit}} : X \mapsto \hom_{\mathcal{C}^{\heartsuit}}(X, D^{\heartsuit}X)$$

$\rightsquigarrow$ $W(\mathcal{C}^{\heartsuit}, \text{Ϙ}^{gs}_{D^{\heartsuit}})$.

From last time, in this situation,

$$L_n(\mathcal{C}, \text{Ϙ}^s_D) = \begin{cases} W(\mathcal{C}^{\heartsuit}, \text{Ϙ}^{gs}_{D^{\heartsuit}}) & n \equiv 0 \mod 4 \\ W(\mathcal{C}, \text{Ϙ}^{gs}_{-D^{\heartsuit}}) & n \equiv 2 \mod 4. \end{cases}$$

0 odd.

~~Goal~~

Suffices to prove devissage for Witt gp's!

$$\left.\begin{aligned} & W\left(\mathrm{Vect}^{fd}(\mathbb{F}_p),\ \mathfrak{I}^{\varphi}_{\mu}\right) \\ \simeq\ & W\left(\mathfrak{S}\text{-tors}_p,\ \mathfrak{I}^{\varphi}_{\Sigma\mu}\right) \end{aligned}\right\}\ \text{classical}$$

$\blacksquare$.