SOSE 2021 SEMINAR: TOPOLOGICAL K-THEORY

Topological K-theory was historically the first example of a *generalized cohomology theory* for spaces. This means that it satisfies all the Eilenberg–Steenrod axioms for homology, except that the K-theory of a point is not concentrated in degree 0. After Grothendieck introduced the K-group K_0 of algebraic varieties in 1957, Atiyah and Hirzebruch quickly realized that an analogous definition in topology was interesting. For X a topological space, they defined $K^0(X)$ to be the group completion of the monoid of isomorphism classes of complex vector bundles over X. Coincidentally, in 1957, Bott had just discovered a surprising periodicity phenomenon in the homotopy groups of the orthogonal and unitary groups, which is now known as *Bott periodicity*. This allowed Atiyah and Hirzebruch to extend K^0 to a full-fledged generalized cohomology theory $K^*(X)$.

The first half of this seminar is dedicated to the definition of topological complex K-theory as a generalized cohomology theory. We will study vector bundles on topological spaces and prove Bott's periodicity theorem. The second half will cover some applications and miscellaneous topics. We will construct the Adams operations and prove that real division algebras only exist in dimensions $1, 2, 4, 8$. We will also introduce the Chern character and the connection with Fredholm operators and index theory.

This seminar will assume some background from algebraic topology (nothing beyond what was covered in Algebraic Topology I) and is a good complement to the class **Algebraic Topology II**.

List of talks

(1) **Introduction**

(2) **Vector bundles I: definitions and constructions** [Hat17, §1.1, §1.2]
Define real and complex vector bundles, and morphisms between them. Show that a n-dimensional vector bundle is trivial iff it has n linearly independent sections. Define direct sums and tensor products of vector bundles, and associated fiber bundles. Discuss several examples (tangent bundles, the canonical line bundle on projective space, etc.). Assuming the homotopy invariance of vector bundles, classify vector bundles on S^k in terms of clutching functions [Hat17, Prop. 1.11, 1.14].

(3) **Vector bundles II: paracompact spaces** [Hat17, §1.1, §1.2 Appendix]
Prove the equivalence of two definitions of paracompactness (existence of locally finite refinements vs. existence of partitions of unity). Sketch the proof that CW complexes are paracompact. Show that a real (resp. complex) vector bundle on a paracompact space admits an inner product (resp. a Hermitian inner product). Show that every subbundle of a vector bundle over a paracompact space admits a complement [Hat17, Prop. 1.3] and that every vector bundle over a compact Hausdorff space is a direct summand of a trivial bundle [Hat17, Prop. 1.4].

(4) **Vector bundles III: the classification theorem** [Hat17, §1.2]
Prove the homotopy invariance of vector bundles on paracompact spaces [Hat17, Thm. 1.6]. Introduce the Grassmannian $G_n(\mathbb{R}^k)$ and the tautological bundle $E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$. Prove the classification theorem $[X, G_n(\mathbb{R}^\infty)] \cong \mathrm{Vect}_n(X)$ for X paracompact [Hat17, Thm. 1.16].

(5) **The Grothendieck ring of vector bundles** [Hat17, §2.1]
Review the notion of group completion of a monoid. For X compact Hausdorff, define $K(X)$ (resp. $KO(X)$) as the group completion of the monoid $\mathrm{Vect}_\mathbb{C}(X)$ (resp. $\mathrm{Vect}_\mathbb{R}(X)$). Show that $K(X) \cong \mathbb{Z} \oplus \tilde{K}(X)$ where $\tilde{K}(X)$ is the set of stable isomorphism classes of vector bundles. Show that the tensor product of vector bundles induces a structure of commutative ring on $K(X)$, for which $\tilde{K}(X)$ is an ideal. Discuss the functoriality in X. Compute $K(S^1)$ and $KO(S^1)$.

(6) **Negative K-groups** [Hat17, §2.2]
For $n \geq 0$ and X compact Hausdorff, define $\tilde{K}^{-n}(X) = \tilde{K}(\Sigma^n X)$. Establish the exact sequence $\tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$ for A closed in X compact Hausdorff [Hat17, Prop. 2.9], and the extension to a long exact sequence of negative K-groups. Define the product $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$. Assuming the "fundamental product theorem", prove Bott periodicity [Hat17, Thm. 2.11], and define the 2-periodic cohomology theory $\tilde{K}^*$ on pairs of compact Hausdorff spaces. Define the unreduced K-groups $K^n(X)$, and show that $K^*(X)$ is a graded-commutative ring [Hat17, Prop. 2.14].

(7) **Bott periodicity** [Hat17, §2.1]
Prove the "fundamental product theorem" $K(X \times S^2) \cong K(X)[H]/(H-1)^2$ [Hat17, Thm. 2.2]

(8) **The splitting principle** [Hat17, §2.3]
Compute the ring $K(\mathbb{C}P^n)$ [Hat17, Prop. 2.24]. Prove the Leray–Hirsch theorem for K-theory [Hat17, Thm. 2.25] and deduce the *splitting principle*.

(9) **Lambda rings** [Knu73], [Hat17, §2.3]
Give an elementary introduction to the notions of pre-λ-ring and λ-ring, following for example [Knu73, Ch. I]. As motivation, explain the operations λ^i on $K(X)$ induced by exterior powers of vector bundles. Use the splitting principle to show that $K(X)$ is a λ-ring [Knu73, Prop. p. 17]. Explain how symmetric functions operate on λ-rings, and define the Adams operations ψ^i.

(10) **Adams operations and the Hopf invariant one theorem** [Hat17, §2.3]
Discuss H-spaces and the relation with real division algebras [Hat17, Lem. 2.17]. Define the Hopf invariant and prove [Hat17, Lemma 2.18]. Prove the Hopf invariant one theorem [Hat17, Thm. 2.19] and the corollary that S^{n-1} has trivial tangent bundle iff $n = 1, 2, 4, 8$ [Hat17, Thm. 2.16].

(11) **The Chern character** [Hat17, §3.1, §4.1]
Give a review of the theory of Chern classes for complex vector bundles [Hat17, Thm. 3.2], including the splitting principle for ordinary cohomology (the complex analogue of [Hat17, Prop. 3.3]). Explain the construction of the Chern character $\mathrm{ch}\colon K(X) \to H^{2*}(X, \mathbb{Q})$ as a ring homomorphism. Prove that ch is a rational isomorphism for X a finite cell complex [Hat17, Prop. 4.5].

(12) **The Atiyah–Jänich theorem** [Ati67, Appendix], [Jän65]
Define Fredholm operators on Hilbert spaces, and the H-space $\mathcal{F}$ of Fredholm operators on a separable complex Hilbert space. Construct the index map $[X, \mathcal{F}] \to K(X)$ for X compact Hausdorff, and show that it is an isomorphism [Ati67, Thm. A1]. Another source is [Jän65] (this theorem was proved independently by Atiyah and Jänich).

(13) **Fredholm complexes** [Seg70]
Discuss the article [Seg70] (taking G to be the trivial group), which generalizes the construction of the index to Fredholm complexes. In particular, for X paracompact and not necessarily compact, the group $K^0(X) := [X, \mathcal{F}]$ can be identified with a certain quotient of the monoid of Fredholm complexes over X [Seg70, Thm. 5.1].

(14) **KO-theory and real Bott periodicity**

References

[Ati67] M. F. Atiyah, *K-theory*, Benjamin, 1967, (Notes by D. W. Anderson)
[Hat17] A. Hatcher, *Vector Bundles and K-Theory*, 2017, `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`
[Jän65] K. Jänich, *Vektorraumbündel und der Raum der Fredholm-Operatoren*, Math. Ann. **161** (1965), pp. 129–142
[Knu73] D. Knutson, *λ-Rings and the Representation Theory of the Symmetric Group*, Lecture Notes in Mathematics, vol. 308, Springer, 1973
[Seg70] G. Segal, *Fredholm complexes*, Quart. J. Math. Oxford **21** (1970), pp. 385–402