Topological K-theory.

History:
- Grothendieck defined the K-theory of algebraic varieties in the 1950's:

$$K(X) = K_0(X) = \mathbb{Z}\left[\begin{matrix}\text{iso. classes of}\\ \text{vector bundles on } X\end{matrix}\right] \Big/ \begin{matrix}[B]=[A]+[C] \text{ for every short exact} \\ \text{sequence } 0\to A\to B\to C\to 0.\end{matrix}$$

- At the same time, Bott proved a remarkable periodicity result:

$U(n)$ = group of unitary $n\times n$ complex matrices

$U = \operatorname{colim}_{n\to\infty} U(n)$.

Then: $\pi_k U = \begin{cases} 0 & \text{if } k \text{ even} \\ \mathbb{Z} & \text{if } k \text{ odd}\end{cases}$. complex Bott periodicity.

- 1959: Atiyah-Hirzebruch:

Grothendieck's definition in a topological setting + Bott periodicity $\Rightarrow$ generalized cohomology theory.

X is a compact Hausdorff space.

$$K(X) = K^0(X) = \mathbb{Z}\left[\begin{matrix}\text{iso. classes of complex}\\ \text{vector bundles on } X\end{matrix}\right] \Big/ \text{same relations as above}$$

Over such X, every SES of vector bundles is split: $B \cong A\oplus C$.

Representability theorem: $\exists$ space $\mathbb{Z}\times BU = \coprod_{n\in\mathbb{Z}} BU$

such that $K(X) \cong [X, \mathbb{Z}\times BU]$ homotopy classes of maps.

For more general X, we define $K(X)$ by this formula.

So we get $K : \mathrm{hTop}^{op} \longrightarrow \mathrm{CRing}$.

For $n\geq 0$,
Define $K^{-n}(X) = \tilde{K}^0(\Sigma^n(X_+)) = [\Sigma^n(X_+), BU]$

so that the suspension isomorphism holds: $\tilde{K}^{-n}(\Sigma X) \cong \tilde{K}^{-n-1}(X)$.

What about K^n for $n\geq 1$? Bott periodicity:

$$K^{-n}(pt) = [S^n, BU] = \pi_n BU = \pi_{n-1} U = \begin{cases}\mathbb{Z} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd}\end{cases}$$

More generally: $K^{-n}(X) \cong K^{-n-2}(X)$, so there are only K^0 and K^{-1}.

$\Rightarrow$ we can extend to positive degrees using this periodicity.

$$K^*(X) = K^{\leq 0}(X)[\beta^{-1}] \qquad \underbrace{\beta}_{\text{generator}} \in K^{-2}(pt) \cong \mathbb{Z}. \quad \text{"Bott element"}$$

Thm (Atiyah-Hirzebruch) $K^*: hTop^{op} \longrightarrow grCRing$ is a generalized cohomology theory.

Remark: Why is $K^0(X)$ not defined using vector bundles when X is not compact-Hausdorff? Because this definition does not have good local-to-global properties:

e.g. $X_\infty = \bigcup_{i \geq 0} X_i$ $\quad$ X_i compact Hausdorff $\quad$ (e.g. $\mathbb{C}P^\infty = \bigcup_{i\geq 0} \mathbb{C}P^i$)

$$\Rightarrow \quad K(X_\infty) \twoheadrightarrow \lim_{i\geq 0} K(X_i) \quad \text{(with kernel } \lim^1_{i \geq 0} K^{-1}(X_i)\text{)}.$$

$\forall$ generalized coh. theory

However: $\quad (\xi_i)_{i\geq 0} \in \lim_{i\geq 0} K(X_i)$

$\xi_i = [V_i] - [W_i]$ $\quad$ it can happen that $\dim V_i, W_i \to \infty$ as $i \to \infty$.

Division algebras & parallelizable spheres.

A smooth n-manifold is called parallelizable if its tangent bundle is trivial, i.e. $\exists$ n linearly independent vector fields.

Q: for which n is S^n parallelizable?

$n=0$: ✓

$n=1$:

$S^1 \subset \mathbb{C}$ is a group such that $z \overset{\mu_w}{\longmapsto} z\cdot w$ is smooth $S^1 \to S^1$.

$$D\mu_w : T_1 S^1 \underset{\leftarrow}{\overset{\cong}{\longrightarrow}} T_w S^1 : D\mu_{w^{-1}}$$

$\forall\, v \in T_1 S^1 \setminus \{0\}$, we get a nonvanishing vector field $w \mapsto (D\mu_w)(v)$.

$n=2$: S^2 is not parallelizable, in fact $\nexists$ a nonvanishing vector field.

$n=3$: $S^3 \subset \mathbb{H}$ (quaternions) $\Rightarrow S^3$ is a group $\Rightarrow TS^3$ is trivial.

$n=4$: not parallelizable:

An easy obstruction to parallelizability is the Euler characteristic:

$$TM \text{ is trivial} \Longrightarrow \chi(M) = 0$$

$$\chi(S^n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even.} \end{cases}$$

$\Rightarrow S^{2n}$ is not parallelizable

$n=5$: ?

$n=7$: $S^7 \subset \mathbb{O}$ octonions $\Longrightarrow S^7$ is a non-associative group

$\Rightarrow TS^7$ is trivial.

In general: if $\exists$ division algebra structure on $\mathbb{R}^n$

(i.e.: an $\mathbb{R}$-bilinear map $b: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $b(-,x)$ & $b(x,-)$ is invertible)

$\Longrightarrow$ S^{n-1} is parallelizable.

<u>Thm</u> (Adams, 1958)

If $n \neq 1,2,4,8$, there does not exist a continuous unital multiplication on S^{n-1}.

also proved at the same time by Bott-Milnor and by Kervaire $\longrightarrow$ In particular, $\nexists$ division algebra structure on $\mathbb{R}^n$.

Adams: "On the non-existence of elements of Hopf invariant one"

$\hookrightarrow$ used "secondary operations" in ordinary cohomology.

Later (1966), Adams–Atiyah gave another proof using topological K-theory:

"K-theory and the Hopf invariant".

$\hookrightarrow$ uses Adams operations in K-theory.

<u>The spectrum KU</u>.

$$\begin{tikzcd}
(\mathrm{Sets}, \times) \arrow[r, shift left, "\mathbb{Z}"] \arrow[d, shift right] & (\text{Abelian groups}, \otimes) \arrow[l, shift left, "\text{forget}"] \arrow[d, shift right] \\
(\mathrm{Spaces}, \times) \arrow[u, shift right, "\pi_0"] \arrow[r, shift left, "\Sigma^\infty_+"] & (\mathrm{Spectra}, \otimes) \arrow[l, shift left, "\Omega^\infty"] \arrow[u, shift right, "\pi_0"]
\end{tikzcd}$$

unit is $\mathbb{Z} = \pi_0 \mathbb{S}$

$\uparrow$

unit is $\mathbb{S} = \operatorname{colim}_{n \to \infty} \Omega^n S^n$

A <u>spectrum</u> is a sequence of pointed spaces $(X_0, X_1, X_2, \ldots)$ and htpy equivalences $X_i \simeq \Omega X_{i+1}$

$\equiv$ a commutative group in Spaces up to coherent homotopy

An abelian group A corresponds to $(A, BA, B^2A, \ldots)$.

On the other hand, Spectra $\equiv$ generalized cohomology theories on CW complexes.

$(X_0, X_1, \ldots) \longmapsto E^*(K)$, $E^n(K) = [K, X_n]$.

$\cup$

Abelian groups $\equiv$ ordinary cohomology theories $H^*(-, A)$.

$K^*(-) \rightsquigarrow$ spectrum $KU = (\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots)$

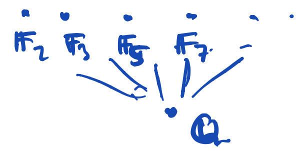
$\Omega(\mathbb{Z} \times BU) = \Omega BU = U$

$\Omega U \simeq \mathbb{Z} \times BU$ (Bott periodicity).

chromatic homotopy theory:

$\mathbb{Z}$ has residue fields $\mathbb{F}_p$ and $\mathbb{Q}$.

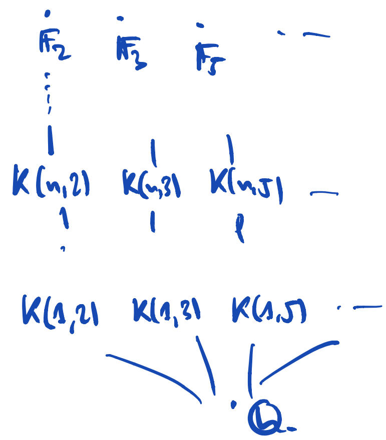
Spec $\mathbb{Z}$:



$A \rightsquigarrow \mathrm{supp}(A) = \{ \kappa \mid A \otimes \kappa \neq 0 \}$

Spec $\mathbb{S}$:

Morava K-theory of height n



$\mathrm{supp}(\mathbb{Z}) = \{ \mathbb{Q}, \mathbb{F}_p \text{ } p \text{ prime} \}$.

$\mathrm{supp}(KU) = \{ \mathbb{Q}, K(1,p) \text{ } p \text{ prime} \}$.