Bott periodicity

X compact Hausdorff, all vector bundles are complex.

Reminder: H tautological line bundle on $\mathbb{CP}^1 \cong S^2$

with clutching function $S^1 \longrightarrow GL_1(\mathbb{C}) \cong \mathbb{C}^*$

$z \longmapsto z$

$(H-1)^2 = 0$ in $K(S^2)$

$\rightsquigarrow \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(S^2)$

Thm (Fundamental product theorem)

The external product map

$$K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\mu} K(X \times S^2)$$

is an isomorphism of rings.

(last time $\Rightarrow \tilde{K}(S^2 X) \cong \tilde{K}(X)$)

Main tool: describe vector bundles on $X \times S^2$ via clutching functions:

Let $p : E \to X$ vector bundle and $f \in \mathrm{Aut}(E \times S^1 \to X \times S^1)$.

$$X \times S^2 \cong X \times D_+ \underset{X \times S^1}{\cup} X \times D_- \rightsquigarrow \underset{E \times \partial D_+ \underset{f}{\cong} E \times \partial D_-}{E \times D_+ \cup E \times D_-} \longrightarrow X \times S^2$$

is a vector bundle over $X \times S^2$, denoted by $[E, f]$.

A homotopy of clutching functions $f \simeq g$ is an automorphism of $\begin{matrix} E \times S^1 \times I \\ \downarrow \\ X \times S^1 \times I \end{matrix}$

$\Rightarrow$ v.b. over $X \times S^2 \times I$ $\Rightarrow$ $[E, f] \cong [E, g]$.

Key point: every v.b. on $X \times S^2$ is of the form $[E, f]$.

(since $X \times D_\pm \to X$ induces a bijection on $\mathrm{Vect}_{\mathbb{C}}$)

$$\Rightarrow \mathrm{Vect}_{\mathbb{C}}(X \times S^2) \cong \Big\{ [E, f] : E \to X \text{ v.b., } f \in \mathrm{Aut}(E \times S^1) \Big\} \Big/ \begin{matrix} \text{isomorphisms in } E \\ \text{\& homotopy of clutching} \\ \text{functions.} \end{matrix}$$

Surjectivity of μ : $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{*} K(X \times S^2)$ (Main part)

Want to show: $[E, f] = a_0 * [H^{\otimes 0}] + a_1 * [H^{\otimes 1}]$ for some $a_i \in K(X)$.

e.g.: $[E, id] = [E] * [H^{\otimes 0}]$

$[E, z] = [E] * [H^{\otimes 1}]$

coordinate on $S^1 \subset \mathbb{C}$

$E \times S^1 \longrightarrow E \times S^1$

$(v, z) \longmapsto (v \cdot z, z)$

Strategy: simplify the clutching function f in several steps:

general continuous $f: E \times S^1 \to E \times S^1$

I ↓

Laurent polynomial $f(v,z) = \sum_{i=-n}^{n} a_i(v) z^i$

$a_i \in End(E)$

$E \times S^1 \longrightarrow E \times S^1$

$(v, z) \longmapsto (\sum_i a_i(v) z^i, z)$

II ↓

affine function $f(v,z) = a(v)z + b(v)$

III ↓

$f = id$ or $f = z$ $\Longrightarrow$ DONE.

step I is analysis

steps II–III are linear algebra.

We fix $E \to X$.

Step I. Theorem I • Every clutching function f for E is homotopic to a Laurent polynomial clutching function $\sum_{i=-n}^{n} a_i z^i$.

• Moreover, two such Laurent polynomials that are homotopic as clutching functions are homotopic through Laurent polynomials.

Idea: Fourier analysis: for a continuous function $f: S^1 \to \mathbb{C}$ one has

$$f(z) \text{ "=" } \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{where} \quad a_n = \int_{S^1} f(z) \bar{z}^n \, dz$$

arc length $/2\pi$

$\langle f, z^n \rangle$ Hermitian product on $L^2(S^1, \mathbb{C})$.

not quite true for arbitrary continuous f, but any f can be uniformly approximated by the Fourier series $\sum_{n=-\infty}^{\infty} a_n r^{|n|} z^n$ as $r \to 1$, $r < 1$.

Lemma Let X compact space, $f: X \times S^1 \to \mathbb{C}$ continuous. Let

$$a_n : X \to \mathbb{C} \qquad a_n(x) = \int_{S^1} f(x,z)\bar{z}^n \, dz$$

Then
$$\underbrace{\sum_{n=-\infty}^{\infty} a_n(x) r^{|n|} z^n}_{u(x,z,r)} \longrightarrow f(x,z) \quad \text{uniformly in } (x,z) \in X \times S^1.$$
as $\underset{r<1}{r \to 1}$

$u(x,z,r)$.

Proof sketch: Consider the Poisson kernel:

$$P(r,z) = \sum_{n=-\infty}^{\infty} r^{|n|} z^n \qquad r \in [0,1), \; t \in S^1.$$

$$= \frac{1-r^2}{(1-rz)(1-r\bar{z})}$$

Then

- $\int_{S^1} P(r,z)\, dz = 1 \quad \forall r.$
- $P(r,z) \to 0$ as $r \to 1$ for $z \neq 1$.

We can write $u(x,t,r) = \int_{S^1} P(r, t\bar{w}) f(x,w)\, dw$

$$|u(x,z,r) - f(x,z)| \leq \int_{S^1} P(r, z\bar{w}) \, |f(x,w) - f(x,z)| \, dw$$

$$\longrightarrow 0 \text{ as } r \to 1$$

uniformly in (x,z) since $X \times S^1$ compact. $\square$.

Proof of Thm I.

$\exists$ Hermitian product on $E \to X$.

$\Rightarrow$ $\mathrm{End}(E)$ is a normed vector space with

$$\|\alpha\| = \sup_{\substack{v \in E \\ |v|=1}} |\alpha(v)|$$

compact.

in particular, locally convex

$\mathrm{Aut}(E) \subset \mathrm{End}(E)$ is open: preimage of $\mathbb{R}_{>0}$ under the cts. map

$$\alpha \longmapsto \inf_{x \in X} |\det(\alpha_x)|$$

We apply this to $E \times S^1 \longrightarrow X \times S^1$.

$\Rightarrow$ it suffices to show that Laurent polynomials are dense in $\mathrm{End}(E \times S^1)$:

$\forall f \in \mathrm{Aut}(E \times S^1)$, $\exists$ convex nbhd B of f in $\mathrm{Aut}(E \times S^1)$

$\exists$ Laurent polynomial $g \in B$

$\Rightarrow$ $tf + (1-t)g$ homotopy from f to g in $\mathrm{Aut}(E \times S^1)$.

$\{U_i\}$ trivializing open cover of $E \xrightarrow{p} X$, $\{\varphi_i\}$ partition of unity, $X_i = \mathrm{supp}(\varphi_i) \subset U_i$.

Over U_i $p^{-1}(U_i) \times S^1 \cong \mathbb{C}^n \times U_i \times S^1 \rightsquigarrow f|_{X_i} = f_i : X_i \times S^1 \longrightarrow GL_n(\mathbb{C})$

By Lemma $f_i \xleftarrow{\|\cdot\|} g_{i,n}$ with Laurent polynomial entries

$f \xleftarrow{\|\cdot\|} \sum_i \varphi_i g_{i,n}$ Laurent polynomials $\square$.

<u>Step II</u>. Let $f = \sum_{i=-n}^{n} a_i z^i$ $a_i \in \mathrm{End}(E)$

$f = z^{-n} \cdot p$ where p polynomial

$$[E,f] \cong \underbrace{\pi_2^*(H^{\otimes -n})}_{\substack{[\mathbb{C} \times X,\, z^{-n}] \\ \in \mathrm{Im}(\mu)}} \otimes [E,p]$$

WLOG, $f = \sum_{i=0}^{n} a_i z^i$

<u>Theorem II</u>. For such f, $[E,f] \oplus [E^{\oplus n}, id] \cong [E^{\oplus n+1}, g]$

where $g(v,z) = a(v)z + b(v)$, $a,b \in \mathrm{End}(E^{\oplus n+1})$.

<u>Pf</u>.

$$g = \left(\begin{array}{cccc|c} 1 & -z & & 0 & 0 \\ & 1 & -z & & \vdots \\ & & 1 & \ddots & 0 \\ 0 & & & 1 & -z \\ \hline a_n & a_{n-1} & \cdots & a_1 & a_0 \end{array}\right) \in \mathrm{Mat}_{n+1}(\mathrm{End}(E \times S^1)) \cong \mathrm{End}(E^{\oplus n+1} \times S^1)$$

$col_{i+1} + z \cdot col_i$ $i = 1, \dots, n$

$$\rightsquigarrow \left(\begin{array}{c|c} I_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline * & f \end{array}\right)$$

$row_{n+1} - *_i \cdot row_i$ $i = 1, \dots, n$

$$id \oplus f = \left(\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \hline 0 & \cdots & 0 & f \end{array}\right) \in \mathrm{Aut}(E^{\oplus n+1} \times S^1).$$

$id \oplus f$ is a clutching function for $[E^{\oplus n}, id] \oplus [E,f]$.

We can go from g to $\mathrm{id} \oplus f$ using elementary operations:

$$e_{ij}(\lambda) = \begin{array}{c} \\ i \end{array}\begin{pmatrix} 1 & & \boxed{\lambda} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$\Longrightarrow \quad g \in \mathrm{Aut}(E^{\oplus n+1} \times S^1)$

$e_{ij}(\lambda)$ is homotopic to I via $e_{ij}(\lambda t)$

$\Longrightarrow \quad \mathrm{id} \oplus f \simeq g$ as clutching function. $\square$

<u>Step III</u> $\quad [E, f], \quad f(v,z) = a(v)z + b(v)$.

<u>Reduction to</u> $a = \mathrm{id}_E$, i.e. $f(v,z) = vz + b(v)$.

$$u(v,z,t) = (1+zt)\left(a(v)\frac{z+t}{1+tz} + b(v)\right) \qquad t=0: \ f(v,z).$$

for $0 \leq t < 1$, $1+zt \in \mathbb{C}^*$ and $\dfrac{z+t}{1+tz} \in S^1$.

$\Longrightarrow$ $u(v,z,t)$ is a homotopy of clutching function from $t=0$ to any $t=t_0<1$,
$[E,f] \cong [E, u(-,t_0)]$

OTOH: $u(v,z,t) = \underbrace{(a(v) + t\,b(v))}_{\substack{\text{automorphism} \\ \text{of } E \text{ for } t=1 \\ \Rightarrow \text{ for all } t \text{ close to } 1.}} z + ta(v) + b(v)$.

For $t_0 < 1$ close to 1: $u(v,z,t_0) = \underset{\in \mathrm{Aut}(E)}{h}\Big(vz + \underbrace{h^{-1}(ta(v) + b(v))}_{b'(v)}\Big)$

$$[E,f] \cong [E, u(-,t_0)] = [E, h(vz + b'(v))]$$
$$\cong [E, vz + b'(v)] \quad . \qquad \square$$

Now: $[E,f]$, $f(v,z) = vz + b(v)$.

$f \in \mathrm{Aut}(E \times S^1) \Longrightarrow b$ has no eigenvalues on S^1

$\left(\begin{array}{l} \text{if } b(v) = \lambda v, \ \lambda \in S^1, v \neq 0 \\ \text{then } f(v, -\lambda) = 0 \end{array}\right)$

Lemma Let V be a f.d. $\mathbb{C}$-vector space, $b: V \to V$ with no eigenvalues on S^1. Then $\exists$ unique decomposition

$$V = V_+ \oplus V_-$$

with $b(V_+) \subset V_+$, $b(V_-) \subset V_-$

and eigenvalues of $b|_{V_+}$ are $|\cdot| > 1$

——— $b|_{V_-}$ — $|\cdot| < 1$.

Pf. $p(T)$ char polynomial of b.

$$p(T) = \underbrace{p_+(T)}_{|\text{roots}|>1} \underbrace{p_-(T)}_{|\text{roots}|<1} \rightsquigarrow V_\pm = \ker(p_\pm(b) : V \to V). \quad \square$$

Apply this fiberwise to $b \in \mathrm{End}(E)$

$$\rightsquigarrow \quad E = E_+ \oplus E_-$$

$$\rightsquigarrow \quad [E, vz + b(v)] \cong [E_+, vz + b_+(v)] \oplus [E_-, vz + b_-(v)]$$

We have homotopies:

$$\left.\begin{array}{l} vzt + b_+(v) \\ vz + b_-(v)t \end{array}\right. \quad \in \mathrm{Aut}(E \times S^1) \text{ for all } t \in [0,1]$$

$$\Rightarrow \quad [E, f] \cong [E_+, \underset{\substack{\uparrow \\ \mathrm{Aut}(E_+)}}{b_+}] \oplus [E_-, z]$$

$$\cong [E_+ ; \mathrm{id}] \oplus [E_-, z]$$

$$= [E_+] * [H^{\otimes 0}] + [E_-] * [H^{\otimes 1}]. \qquad \square$$