REGENSBURG RESEARCH SEMINAR SS 2022

TEMPERED COHOMOLOGY AND EQUIVARIANT ELLIPTIC COHOMOLOGY

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Introduction

Tempered cohomology is a global equivariant cohomology theory associated with a p-divisible group over an $\mathbb{E}_\infty$-ring, which provides higher-chromatic generalizations of equivariant topological K-theory. It is a unifying framework connecting all of the following concepts: p-divisible groups and formal groups, global equivariant homotopy theory, the Atiyah–Segal completion theorem, and the classical and chromatic character theory of finite groups.

The goal of this seminar is to learn the basics of tempered cohomology following Jacob Lurie's *Elliptic Cohomology III*. We will also review some of the theory of spectral elliptic curves (whose associated tempered cohomology is equivariant elliptic cohomology) and Lurie's construction of the $\mathbb{E}_\infty$-ring of topological modular forms. Some working knowledge of higher category theory and $\mathbb{E}_\infty$-ring spectra will be assumed.

The main reference will be the series of papers [Lur17], [Lur18a], [Lur19], in particular the latter two. A useful bird's eye view can be found in the survey [Lur09].

1. **Introduction (03.05)**

Part I: Formal groups and elliptic cohomology

2. **Spectral algebraic geometry (10.05)** Define a spectral Deligne-Mumford stack [Lur18b, Df. 1.4.4.2] and strict abelian varieties [Lur17, Df. 1.5.1]. State the Artin representability theorem [Lur18b, Th. 18.3.0.1] and use it to prove the existence of the moduli stack of strict abelian varieties [Lur17, Th. 2.4.1].
3. **Spectral formal groups (17.05)** Define what is a smooth coalgebra over an E_∞-ring [Lur18a, Df. 1.2.4] and prove [Lur18a, Pr. 1.5.9]. Use it to define spectral formal groups [Lur18a, Df. 1.6.1]. Construct the formal multiplicative group [Lur18a, Cn. 1.6.16] and the formal group of a strict abelian variety [Lur18a, Cn. 7.1.1].
4. **Orientations and Quillen formal groups (24.05)** Construct the Quillen formal group of an even periodic E_∞-ring [Lur18a, Cn. 4.1.13]. Define preorientations and orientations of a formal hyperplane [Lur18a, Df. 4.3.1 and Df. 4.3.9] and prove the universal property of the Quillen formal group law [Lur18a, Pr. 4.3.23].
5. **Barsotti-Tate groups (31.05)** Define a p-divisible group [Lur19, Df. 2.1.1] and $\mathbf{P}$-divisible group. State the existence of the identity component of a p-divisible group over a p-complete E_∞-ring [Lur18a, Th. 2.0.8]. Define a preorientation [Lur19, Df. 2.6.8] and prove that giving a preorientation of a $\mathbf{P}$-divisible group is equivalent to giving a preorientation of the identity component of the p-completion for every p [Lur19, Rm. 2.6.9 and Pr. 2.2.1]. Give the definition of an orientation [Lur19, Df. 2.6.12]. Give the examples

of the multiplicative group [Lur19, Pr. 2.8.2] and of the P-divisible group of an elliptic curve [Lur19, Pr. 2.9.2].

6. **The construction of elliptic cohomology (07.06)** Define and prove the existence of the stack of oriented elliptic curves [Lur18a, Pr. 7.2.10]. Sketch how this can be used to prove the Goerss-Hopkins-Miller theorem [Lur18a, Th. 7.0.1]

Part II: The construction of tempered cohomology

7. **Orbispaces and equivariant homotopy theory (14.06)** Define the global orbit category $\mathscr{T}$ [Lur19, Nt. 3.1.1] and orbispaces [Lur19, Df. 3.1.4]. Explain the relationship between orbispaces and genuine G-spaces for a group G [Lur19, Cn. 3.2.16, Ex. 3.2.18, Pr. 3.3.13]. Construct the formal loop space of an orbispace [Lur19, Cn. 3.4.3].
8. **The tempered cohomology associated to a preorientation (21.06)** Characterize preorientations in term of extensions of the **P**-divisible group to $\mathscr{T}$ [Lur19, Theorem 3.5.5] and use it to define the tempered cohomology associated to a **P**-divisible group ([Lur19, Cn. 4.0.3 and 4.0.5])

Part III: Properties of tempered cohomology

9. **Equivariant K-theory as tempered cohomology (28.06)** Recall the classical definition of equivariant K-theory in term of vector bundles [Lur19, Cn. 3.6.1 and Nt. 3.6.5]. Show that it coincides with the tempered cohomology of μ_∞ [Lur19, Cor. 3.6.10 and Th. 4.1.2].
10. **The Atiyah-Segal comparison map (05.07)** Construct the Atiyah-Segal comparison map [Lur19, Co. 4.2.2]. Prove the Atiyah-Segal completion theorem for the action of an abelian group on the point [Lur19, Pr. 4.2.8].
11. **The character map for tempered cohomology (12.07)** Construct the character map for tempered cohomology [Lur19, Th. 4.3.2 and Nt. 4.3.3]. Explain how this realizes to the classical Chern character in the case of K-theory [Lur19, Ex. 4.3.8 and 4.3.9]. State the basechange theorem for tempered cohomology [Lur19, Th. 4.7.1] and deduce from that the character isomorphism [Lur19, Cr. 4.7.7]
12. **The Atiyah-Segal completion theorem (19.07)** Explain the needed finiteness results for tempered cohomology [Lur19, Cr 4.7.13] and give the proof of the tempered Atiyah-Segal completion theorem [Lur19, Th. 4.5.2]

References

[Lur09] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Springer, 2009, pp. 219–277, `https://www.math.ias.edu/~lurie/papers/survey.pdf`

[Lur17] ______, *Elliptic cohomology I: Spectral Abelian Varieties*, 2017

[Lur18a] ______, *Elliptic cohomology II: Orientations*, 2018

[Lur18b] ______, *Spectral Algebraic Geometry*, 2018

[Lur19] ______, *Elliptic cohomology III: Tempered cohomology*, 2019