1. Complex oriented cohomology theories

Let R be a homotopy commutative ring spectrum. We say that R is **complex orientable** if there exists a factorization

$$\begin{array}{ccc} \mathbb{S} \simeq \Sigma^{-2}\Sigma^\infty \mathbb{CP}^1 & \xrightarrow{\quad 1 \quad} & R \\ & \searrow \qquad \nearrow_{\bar{u}} & \\ & \Sigma^{-2}\Sigma^\infty \mathbb{CP}^\infty & \end{array}$$

We call such a factorization $\bar{u} \in \widetilde{R}^2(\mathbb{CP}^\infty)$ a **complex orientation**. A complex orientation is exactly the datum needed to define a Thom class ("orientation") for each complex vector bundle. In particular we can define a theory of Chern classes. Today we will be only interested in c_1 of line bundles. If L is a line bundle over a space X, it is classified by some map $f_L : X \to BU_1 \simeq \mathbb{CP}^\infty$. Therefore we can define

$$c_1(L) := f_L^* u\,.$$

Example 1. *If $R = \mathbb{Z}$, we can take $u \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ be the class corresponding to the 2-cell in the standard decomposition. Then $c_1(L)$ is the usual Chern class.*

Example 2. *If $R = \mathrm{KU}$ is the complex K-theory spectrum, we can take $u \in \widetilde{\mathrm{KU}}^2(\mathbb{CP}^\infty)$ given by $([\eta] - 1)\beta^{-1}$ where η is the tautological bundle and β is the Bott class. Then $c_1(L) = ([L] - 1)\beta^{-1}$.*

Example 3. *If $R = \mathrm{MU} := \mathrm{Th}(BU)$ is the complex cobordism spectrum, the equivalence $\mathrm{Th}(BU_1) \simeq \Sigma^\infty \mathbb{CP}^\infty$ identifies the map $\mathrm{Th}(BU_1) \to \mathrm{Th}(BU) = \mathrm{MU}$ with a complex orientation.*

If R is complex orientable, the Atiyah-Hirzebruch spectral sequence degenerates and one gets isomorphisms

$$R^*(\mathbb{CP}^\infty) \simeq R_*[[u]] \qquad R^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq R_*[[u_1, u_2]]$$

So, if we take the map $\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifying the sum of line bundles we obtain $\mu^* u = F(u_1, u_2)$ where F is a power series such that

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))\,.$$

Example 4. *In the case $R = \mathbb{Z}$ we have $F(x, y) = x + y$. In the case $R = \mathrm{KU}$ we have $F(x, y) = \beta^{-1}[(1 + \beta x)(1 + \beta y) - 1]$*

The power series F satisfies the following properties

1. $F(x, 0) = F(0, x) = x$;
2. $F(x, F(y, z)) = F(F(x, y), z)$;
3. The coefficient $a_{ij} \in \pi_* R$ of $x^i y^j$ lies in degree $2(i + j) - 2$

If A_* is a graded ring a power series $F \in A[[x, y]]$ with the above three properties is called a graded formal group law. There is a universal graded formal group law over the graded ring

$$\mathbb{L} \to \mathbb{Z}[a_{ij}]/(--)$$

Therefore for every complex oriented homotopy ring spectrum R we have a map $\mathbb{L} \to \pi_* R$.

Theorem 5 (Quillen)**.** *The map*

$$\mathbb{L} \to \pi_* \mathrm{MU}$$

is an isomorphism.

So for any complex oriented cohomology theory we obtain a graded formal group law. Can we go the other way around? Sometimes! Let $[n] = F(x, [n-1](x))$ be the "multiplication by n" power series.

Theorem 6 (Landweber exact functor theorem)**.** *Suppose (A_*, F) is a graded formal group law and for every prime p let v_n be the coefficient of x^{p^n} in $[p]$. Then the functor*

$$X \mapsto \mathrm{MU}_* X \otimes_{\mathrm{MU}_*} A_*$$

is a homology theory if $(p, v_1, v_2, \dots)$ is a regular sequence. In which case the homotopy ring spectrum representing it is the unique homotopy ring spectrum with the given graded formal group law.

Example 7. *Let $A_* = \mathbb{Z}[\beta^{\pm 1}]$ with $|\beta| = 2$ and $F(x,y) = x + y + \beta xy$. Then this graded formal group law is Landweber exact and the corresponding homotopy ring spectrum is* KU*. This is the famous Conner-Floyd theorem.*

We will be interested in a special kind of ring spectra. We say that R is **weakly even periodic** if the maps

$$\pi_2 R \otimes_{\pi_0 R} \pi_n R \to \pi_{n+2} R$$

are isomorphisms for every $n \in \mathbb{Z}$. Note that by choosing $n = -2$ this implies that $\pi_2 R$ is an invertible module and $\pi_{2n} R \simeq (\pi_2 R)^{\otimes n}$. We say that R is **complex periodic** if it is complex orientable and weakly even periodic. The prototypical example is KU. The graded formal group laws appearing in these case are special

A **formal group** over a commutative ring R is a pair (ω, F) where ω is an invertible R-module and F is a graded formal group law over $\overline{\mathrm{Sym}}_* \omega := \bigoplus_{n \in \mathbb{Z}} \omega^{\otimes n}$. It is clear that if A is a complex periodic ring spectrum, then we obtain a formal group by taking $(\pi_2 A, F)$.

Algebraic geometry is a good source of formal groups.

Example 8. *Let R be a commutative ring and let $\mathbb{G}$ be a locally of finite presentation, geometrically connected, geometrically reduced flat purely 1-dimensional group scheme over R (from now one "group scheme over R"). Then we can get a formal group (ω, F) by taking ω to be the cotangent bundle of $\mathbb{G}$ at the indentity and by F the Taylor series of the multiplication operation $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$.*

If $\mathbb{G} = \mathbb{G}_m$ Is the multiplicative group, we obtain exactly the formal group of K-theory (since the cotangent bundle of $\mathbb{G}_m$ is canonically trivialized). The other important example comes from elliptic curves. An **elliptic curve** $E \to \operatorname{Spec} R$ is a proper group scheme over R.

Definition 9. *An elliptic cohomology theory is a triple (E, A, φ) where $E \to \operatorname{Spec} R$ is an elliptic curve, A is a complex periodic ring spectrum and φ is an isomorphism between the formal group of E and the formal group of A.*

If the formal group of E is Landweber exact, then we can always build a unique elliptic cohomology theory. We want to study the general case, and possibly do it in a more structured way.

Let $\mathcal{M}_{ell}$ be the moduli stack of elliptic curves, that is the Deligne-Mumford stack such that $\mathrm{Map}(\mathrm{Spec}\, R, \mathcal{M}_{ell})$ is naturally equivalent to the groupoid of elliptic curves over R. Then it's easy to check that every elliptic curve classified by a flat map $\mathrm{Spec}\, R \to \mathcal{M}_{ell}$ is Landweber exact, and therefore gives rise to an elliptic cohomology theory.

Theorem 10 (Goerss-Hopkins-Miller)**.** *There is a sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on the étale topos of $\mathcal{M}_{ell}$ such that for every flat $E : \mathrm{Spec}\, R \to \mathcal{M}_{ell}$ the E_∞-ring $\Gamma(\mathrm{Spec}\, R, \mathcal{O}^{top})$ represents the elliptic cohomology theory associated to E.*

Given the Goerss-Hopkins-Miller theorem we can define the elliptic cohomology theory associated to any elliptic curve $E : \mathrm{Spec}\, R \to \mathcal{M}_{ell}$ as $\Gamma(\mathrm{Spec}\, R, \mathcal{O}^{top})$. Moreover we can consider also $\Gamma(\mathcal{M}_{ell}, \mathcal{O}^{top})$ (note that this is not strictly speaking an elliptic cohomology theory). This is an E_∞-ring spectrum known as TMF.

The original proof the Goerss-Hopkins-Miller theorem went through a complicated obstruction theory argument, which moreover proved that there exists a unique such sheaf of E_∞-ring spectra. In this seminar we will do a more conceptual proof based on spectral algebraic geometry.

2. Spectral algebraic geometry

To prove the Goerss-Hopkins theorem, we will use ideas from spectral algebraic geometry. For now we will not enter into the details of the definitions (that's what the next talk is for!), rather let us take for granted that there is a notion of algebraic geometry over an E_∞-ring. In particular we will use that there are subcategories of $\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Spc})$ whose objects we will call spectral schemes and spectral Deligne-Mumford stacks respectively.

A (strict) commutative group scheme over R is a functor

$$\mathbb{G} : \mathrm{CAlg}_R \to \mathcal{D}(\mathbb{Z})_{\geq 0}$$

such that it becomes representable by a spectral scheme over R after postcomposing with $\mathcal{D}(\mathbb{Z})_{\geq 0} \to \mathrm{Spc}$.

Example 11. *The multiplicative group $\mathbb{G}_m : \mathrm{CAlg}_R \to \mathcal{D}(\mathbb{Z})_{\geq 0}$ is defined by sending S to the commutative group of* strict *units of S:*

$$\mathbb{G}_m(S) := t_{\geq 0}\, \mathrm{map}_{\mathrm{Sp}}(\mathbb{Z}, \mathrm{gl}_1(R))$$

where $\mathrm{gl}_1(R)$ is the spectrum of invertible elements of R. Note that we need this trick to make it into a $\mathbb{Z}$-module.

Example 12. *A spectral elliptic curve over R is a proper, locally of almost finite presentation geometrically reduced, geometrically connected commutative group scheme of dimension 1.*

A preorientation over a commutative group scheme $\mathbb{G}$ is a map of $\mathbb{Z}$-modules $\mathbb{Z}[2] \to \mathbb{G}(R)$ or, equivalently, a class $u \in \pi_2 G(R)$.

Example 13. *Let $\mathbb{G}_m$ be the multiplicative group. Then the space of preorientations of $\mathbb{G}_m$ over R is exactly*

$$\mathrm{Map}_{\mathbb{Z}}(\mathbb{Z}[2], \mathbb{G}_m(R)) \simeq \mathrm{Map}_{E_\infty-Grp}(B^2\mathbb{Z}, \mathrm{GL}_1(R)) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{S}[\mathbb{CP}^\infty], R)$$

Note that if $\mathbb{G}$ is a group scheme over R we have that $\Omega\mathbb{G}$ is affine and represented by $R \otimes_{\mathcal{O}_{\mathbb{G},e}} R$, therefore $\Omega^2\mathbb{G}$ is also affine represented by $\mathcal{O}_{\mathbb{G},e}^{\otimes S^2} \otimes_{\mathcal{O}_{\mathbb{G},e}} R$. Thus the space of preorientations is always representable by an E_∞-ring R'. Moreover a simple computation shows $\pi_0(R) \to \pi_0(R')$ is an isomorphism if R is connective.

We can then consider the stack $\mathcal{M}_{ell}^{pre}$ of preoriented (spectral) elliptic curves. Then underlying ∞-topos of $\mathcal{M}_{ell}^{pre}$ is equivalent to the ∞-topos of $\mathcal{M}_{ell}$, precisely we can see the map $\mathcal{M}_{ell} \to \mathcal{M}_{ell}^{pre}$ as a "nilpotent thickening". Thus we can think of $\mathcal{M}_{ell}^{pre}$ as the datum of a connective sheaf of E_∞-rings $\mathcal{O}'$ on $\mathcal{M}_{ell}$. One can think of $\mathcal{O}'(R)$ as the smallest E_∞-ring where we can define a lift of E. Unfortunately $\mathcal{O}'$ is not quite what we want since, for example, it is not complex periodic. But we can fix this.

The dualizing line $\omega_{\mathbb{G}}$ of an algebraic group $\mathbb{G}$ is the unique R-module sitting in the fiber sequence

$$\Sigma\omega_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G},e} \otimes_R \mathcal{O}_{\mathbb{G},e} \to \mathcal{O}_{\mathbb{G},e}$$

(that is $\omega_{\mathbb{G}}$ is the ideal of the diagonal in $\mathbb{G} \times \mathbb{G}$ at the identity section). Then for every preorientation α we can define the Bott map

$$\Sigma^2\omega_{\mathbb{G}} \to \Sigma(\mathcal{O}_{\mathbb{G},e} \otimes_R \mathcal{O}_{\mathbb{G},e}) \xrightarrow{\alpha} R$$

We say that α is an orientation when this map is an equivalence.

I cannot find a direct proof, but a preorientation is just a map of formal group $\operatorname{Spf} R^{\mathbb{CP}^\infty} \to \widehat{\mathbb{G}}$, and it is an orientation exactly when it is an equivalence.

Example 14. *In the case $\mathbb{G} = \mathbb{G}_m$ we have $\omega_{\mathbb{G}_m} \simeq \mathbb{S}$ canonically. Moreover for the universal preorientation over $\mathbb{S}[\mathbb{CP}^\infty]$ we have that the Bott map $\Sigma^2\mathbb{S}[\mathbb{CP}^\infty] \to \mathbb{S}[\mathbb{CP}^\infty]$ is exactly multiplication by the Bott element $\beta \in \pi_2\mathbb{S}[\mathbb{CP}^\infty]$. Therefore the space of orientations of $\mathbb{G}_m$ is represented by $\mathbb{S}[\mathbb{CP}^\infty][\beta^{-1}]$.*

Using the fact that $\omega_{\mathbb{G}}$ is projective of rank 1, it is easy to show that the space of orientations of an algebraic group is always representable. We write $\mathfrak{O}_{\mathbb{G}}$ for the E_∞-ring over R classifying the space of orientations

Theorem 15 (Snaith)**.** *The E_∞-ring $\mathfrak{O}_{\mathbb{G}_m}$ is equivalent to* KU*.*

One could now ask what happens if we consider $\mathfrak{O}_E$ for E an elliptic curve over a discrete ring. Unfortunately the answer is "not much interesting": if $\mathbb{G}$ is an algebraic group over a discrete ring $\mathfrak{O}_{\mathbb{G}}$ is always an algebra over the rational numbers.

We will deduce the Goerss-Hopkins-Miller theorem from the analog statement for elliptic cohomology. In order to do so we need first to find a lift of the universal elliptic curve on $\mathcal{M}_{ell}$ over "the sphere spectrum".

Proposition 16. *The functor sending R to the groupoid of spectral elliptic curves over R is represented by a spectral Deligne-Mumford stack $\mathcal{M}_{ell}^s$ whose underlying ∞-topos is the same as the moduli stack of elliptic curves.*

Note that while this gives a sheaf $\mathcal{O}^s$ of E_∞-rings over $\mathcal{M}_{ell}$, it is not the sheaf we are looking for. For example the sections of $\mathcal{O}^s$ over affines are connective and will not be complex orientable (at least in general). One should think about it as a "spectral thickening" of $\mathcal{M}_{ell}$ similar in spirit to $\mathbb{S}$ seen as a spectral thickening of $\mathbb{Z}$. One maybe would like a more explicit description of $\mathcal{M}_{ell}^s$ (similar to the global quotient description of $\mathcal{M}_{ell}$ given by the Weierstraß equations), but as far as I know this has not been developed yet.

Theorem 17 (Lurie)**.** *Let $E : \operatorname{Spec} R \to \mathcal{M}_{ell}$ be an étale map and let E' be the corresponding spectral elliptic curve over $\mathcal{O}^s(E)$. Then the E_∞-ring $\mathfrak{O}_{E'}$ classifying orientations of E' represents an elliptic cohomology theory associated to E.*

Proof. This is essentially a computation of the homotopy groups of $\mathfrak{O}_E$, since we already know that its formal group is equivalent to the formal group associated to E. □

From now on we will consider $\mathfrak{O}_{E'}$ to be "the" elliptic cohomology associated to E.

3. Tempered cohomology

There's an ∞-category Glo whose objects are compact (abelian) Lie groups and whose morphisms are given by

$$\mathrm{Glo}(G, H) = \mathrm{Map}_{Lie}(G, H)_{hH}$$

It receives a map from the ∞-category of compact (abelian) Lie groups (in fact it is a left fibration represented by the trivial group).

For every Lie group G we get a map

$$\mathbf{O}_G^{ab} \to \mathrm{Glo} \qquad G/H \mapsto H$$

where $\mathbf{O}_G^{ab}$ is the subcategory of those orbits with abelian stabilizers. Therefore we have a left Kan extension

$$\mathrm{Spc}_G \to \mathrm{Spc}_{gl}$$

In a sense a cohomology theory on Spc_{gl} is a cohomology theory on Spc_G for every G. For more about it, stay tuned :).

Let $\mathbb{G}$ be a 1-dimensional commutative group scheme. Then for every finitely generated abelian group B there is a group scheme $\mathbb{G}[B]$ such that $\mathbb{G}[B](R) = \mathrm{Map}_{\mathbb{Z}}(B, \mathbb{G}(R))$ (the proof is easy: B is generated by finite colimits by $\mathbb{Z}$ and group schemes have finite limits).

Theorem 18. *The datum of an extension of the functor $\mathbb{G}[-] : \mathrm{Ab}_{fg}^{\mathrm{op}} \to \mathrm{GrpSch}$ along $\widehat{(-)}\ \mathrm{Ab}_{fg} \to \mathrm{Glo}^{\mathrm{op}}$ is exactly the datum of a preorientation of $\mathbb{G}$.*

Therefore if $(\mathbb{G}, \alpha)$ is a preoriented group scheme we obtain a functor

$$\mathrm{Glo} \to \mathrm{GrpSch} \xrightarrow{\Gamma} \mathrm{CAlg}^{\mathrm{op}}$$

By right Kan extending we obtain finally a functor

$$A_{\mathbb{G}}(-) : \mathrm{Spc}_{gl}^{\mathrm{op}} \to \mathrm{CAlg}$$

which is called the **tempered cohomology** of $\mathbb{G}$.

Example 19. *Let $\mathbb{G} = \mathbb{G}_m$ be the oriented multiplicative group. Then the composite*

$$\mathrm{Spc}_G^{\mathrm{op}} \to \mathrm{Spc}_{gl}^{\mathrm{op}} \to \mathrm{CAlg}$$

is given by the equivariant K-theory functor $X \mapsto \mathrm{KU}_G(X)$. So the tempered cohomology of the multiplicative group recovers equivariant K-theory. More generally $A_{\mathbb{G}}$ is represented by the global K-theory spectrum.

We know that KU_G has some interesting properties. For example

Theorem 20 (Atiyah-Segal)**.** *Let G be a compact Lie group and X a finite G-space. Then the map*

$$\pi_* \operatorname{KU}_G(X) \to \pi_* \operatorname{KU}(X_{hG})$$

exhibits the right hand side as the completion of the left hand side at the augmentation ideal.

Theorem 21 (Character theory)**.** *Let G be a finite group and X a finite G-space. Then there is an equivalence*

$$\mathbb{C} \otimes \operatorname{KU}^0_G(X) \simeq H^{2*}\left((\coprod_{g \in G} X^g)/G; \mathbb{C} \right)$$

We will see that these two theorems are true in the case of a tempered cohomology for a general Barsotti-Tate group (where $\mathbb{C}$ will need to be replaced by some ring where the given Barsotti-Tate group is étale). For example this will produce the HKR character theory as a biproduct.