1. Étale maps of E_∞-rings

Let R be an E_∞-ring. Recall that an R-module M is **flat** if $\pi_0 M$ is a flat $\pi_0 R$-module and the map

$$\pi_0 M \otimes_{\pi_0 R} \pi_n R \to \pi_n M$$

is an equivalence for every $n \in \mathbb{Z}$. When R is connective, this is equivalent to the functor $M \otimes -$ being left t-exact.

A map of E_∞-rings $R \to S$ is **étale** if $\pi_0 R \to \pi_0 S$ is an étale map of ordinary rings and S is flat as an R-module.

Theorem 1. *Let R be an E_∞-ring. Then the functor π_0 induces an equivalence*

$$\text{Ét}_R \to \text{Ét}_{\pi_0 R}$$

2. ∞-topoi

An ∞**-topos** is an ∞-category $\mathcal{X}$ that is a left exact localization of a presheaf category. Precisely, there exists a small ∞-category $\mathcal{I}$ and an adjunction

$$\mathcal{X} \underset{i}{\overset{L}{\leftrightarrows}} \mathrm{PSh}(\mathcal{I})$$

where L is left-exact (i.e. it commutes with finite limits) and i is fully faithful.

Example 2. *Suppose τ is a Grothendieck topology on I. Then $\mathcal{X} = \mathrm{Sh}_\tau(I)$ is an ∞-topos (since sheafification is left-exact). It is an open problem whether every ∞-topos can be realized this way.*

Remark 3. *There are in fact intrinsic characterizations of ∞-topoi. For example $\mathcal{X}$ is an ∞-topos iff it is presentable and the functor $\mathcal{X}^{\mathrm{op}} \to \mathrm{Cat}$ sending U to $\mathcal{X}_{/U}$ preserves small limits.*

A map of ∞-topoi ("geometric morphism") is an adjunction

$$\mathcal{X} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{X}'$$

where f^* is left exact. It is easy to see that a morphism of sites induces a morphism of associated topoi.

Example 4. *Suppose $\mathcal{X}$ is an ∞-topos and $U \in \mathcal{X}$. Then $\mathcal{X}_{/U}$ is also an ∞-topos and there is a map of ∞-topoi*

$$\mathcal{X}_{/U} \underset{p_*}{\overset{p^*}{\leftrightarrows}} \mathcal{X}$$

with $p^*(V) = U \times V$.

If $\mathcal{X}$ is an ∞-topos and $\mathcal{C}$ is an ∞-category with small limits, a $\mathcal{C}$-valued sheaf is a limit-preserving functor $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$. The category of $\mathcal{C}$-valued sheaves is denoted $\mathrm{Sh}(\mathcal{X};\mathcal{C})$. A morphism of ∞-topoi $f : \mathcal{X} \to \mathcal{X}'$ induces an adjunction

$$\mathrm{Sh}(\mathcal{X};\mathcal{C}) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Sh}(\mathcal{X}';\mathcal{C})$$

Example 5. *Suppose $\mathcal{X} = \mathrm{Sh}_\tau(\mathcal{I})$. Then using the universal property of Bousfield localizations and of presheaf categories we see that $\mathrm{Sh}(\mathcal{X};\mathcal{C}) \simeq \mathrm{Sh}_\tau(\mathcal{I};\mathcal{C})$.*

Example 6. *By the adjoint functor theorem we have $\mathrm{Sh}(\mathcal{X};\mathrm{Spc}) \simeq \mathcal{X}$. Moreover $\mathcal{X}^\heartsuit := \mathrm{Sh}(\mathcal{X};\mathrm{Set})$ is the full subcategory of discrete objects.*

Now suppose $F \in \mathrm{Sh}(\mathcal{X};\mathrm{Sp})$ is a sheaf of spectra. Then we can define $\pi_n F$ by taking the "sheafification" of $\mathcal{X}^{\mathrm{op}} \xrightarrow{F} \mathrm{Sp} \xrightarrow{\pi_n} \mathrm{Ab}$, that is the best limit-preserving approximation. We say that F is connective if $\pi_n F = 0$ for all $n < 0$.

A map $f : X \to Y$ in an ∞-topos is an effective epimorphism if Y is the limit of its Čech cover. If $X \in \mathcal{X}$ is an object a cover of X is a set of objects $\{U_i\} \in \mathcal{X}_{/X}$ such that $\coprod_i U_i \to X$ is an effective epimorphism. A cover of $\mathcal{X}$ is simply a cover of its terminal object.

3. Spectral Deligne-Mumford stacks

Let R be an E_∞-ring. Then we can take the ∞-topos $\operatorname{Spec} R$ which is given by considering the site $\acute{\mathrm{E}}\mathrm{t}_R$ with the étale topology. It comes with a sheaf of E_∞-rings $\mathcal{O}_{\operatorname{Spec} R}$ sending $R \to S$ to S.

A **(nonconnective) spectral Deligne-Mumford stack** is a pair $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}$ is a sheaf of E_∞-rings on $\mathcal{X}$, such that there exists a cover $\{U_i\}$ of $\mathcal{X}$ and E_∞-rings R_i such that $(\mathcal{X}_{/U_i}, \mathcal{O})$ is equivalent to $(\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$ for every i.

We say that $(\mathcal{X}, \mathcal{O})$ is connective if $\mathcal{O}$ is.

Example 7. *Let X be an ordinary scheme (or even an ordinary Deligne-Mumford stack). Then if we let $X_{\acute{e}t}$ be the étale topos of X, then $(X_{\acute{e}t}, \mathcal{O})$ is a spectral Deligne-Mumford stack. This in fact gives an embedding of classical Deligne-Mumford stacks into spectral Deligne-Mumford stacks.*

The inclusion of classical Deligne-Mumford stacks into all connective spectral Deligne-Mumford stacks has a left adjoint, which we call the "underlying" classical DM-stack (concretely this replaces $\mathcal{O}$ by $\pi_0\mathcal{O}$ and the ∞-topos by its 1-localic approximation).

There is a functor from spectral DM stacks and functors $\mathrm{CAlg} \to \mathrm{Spc}$ sending $\mathcal{X}$ to $\mathrm{Map}(\operatorname{Spec} -, \mathcal{X})$. This is fully faithful. Moreover a spectral DM stack is connective iff this functor factors through the localization $\mathrm{CAlg} \to \mathrm{CAlg}^{cn}$.

Lemma 8. *The space of spectral DM-stacks flat over R is equivalent to the space of spectral DM-stacks flat over $t_{\geqslant 0}R$.*

Proof. The inverse functor is given by the connective cover functor $(\mathcal{X}, \mathcal{O}) \mapsto (\mathcal{X}, t_{\geqslant 0}\mathcal{O})$. □

4. Spectral elliptic curves

We let $\mathrm{Var}_+(R)$ to be the ∞-category of flat spectral DM stacks over R such that $t_{\geqslant 0}X \to \operatorname{Spec} t_{\geqslant 0}R$ is proper, locally almost of finite presentation, geometrically reduced and geometrically connected. Note that the map $\mathrm{Var}_+(t_{\geqslant 0}R) \to \mathrm{Var}_+(R)$ is an equivalence. A strict abelian variety over R is a functor

$$\mathrm{CAlg}_R \to \mathrm{Mod}_{\mathbb{Z}}^{cn}$$

such that it becomes representable by an object in Var_+ after forgetting to spaces. An elliptic curve over R is an abelian variety over R of dimension 1 (that is, such that for every map $t_{\geqslant 0}R \to k$ where k is an algebraically closed field the classical abelian variety over k has dimension 1).

When R is a discrete ring, this reduces to the classical notion of

5. The moduli stack of elliptic curves

Our goal is the following

Theorem 9. *There exists a spectral Deligne-Mumford stack $\mathcal{M}^s_{ell}$ such that for every E_∞-ring R the space $\mathrm{Map}(\mathrm{Spec}\, R, \mathcal{M}^s_{ell})$ is equivalent to the space of strict spectral elliptic curves over R. Moreover the underlying ∞-topos of $\mathcal{M}^s_{ell}$ is equivalent to the ∞-topos of its underlying Deligne-Mumford stack $\mathcal{M}_{ell}$.*

To prove it we will use the following

Theorem 10 (Artin-Lurie representability theorem)**.** *Let $F : \mathrm{CAlg}^{cn} \to \mathrm{Spc}$ be a functor. Suppose that*

- *F is a sheaf for the étale topology;*
- *The restriction of F to discrete rings is represented by a Deligne-Mumford stack $\mathcal{X}_0$;*
- *F is locally almost of finite presentation: it commutes with all filtered limits in $\mathrm{CAlg}_{\leqslant n}$ for every $n < \infty$;*
- *F is nilcomplete: For every R we have $F(R) \simeq \lim F(t_{\leqslant n} R)$;*
- *F is infinitesimally cohesive: for every maps $R \to S$ and $R' \to S$ such that they are surjections with nilpotent kernels on π_0 we have $F(R \times_S R') \simeq F(R) \times_{F(S)} F(R')$*
- *F admits a cotangent complex: for every R and $\eta \in F(R)$ there's an R-module $\mathbb{L}_{F,\eta}$ such that $\mathrm{Map}_R(\mathbb{L}_{F,\eta}, M)$ is equivalent to the fiber of $F(R \oplus M) \to F(R)$. Moreover for every map of E_∞-rings $f : R \to S$ the map $f^*\mathbb{L}_{F,\eta} \to \mathbb{L}_{F,f\eta}$ is an equivalence.*

Then there exists a (connective) spectral Deligne-Mumford stack $\mathcal{X}$ representing F and moreover its ∞-topos is the same as its underlying classical DM stack.

In fact we won't prove most of this (sorry!)

Proposition 11. *The functor $R \mapsto \iota\,\mathrm{Var}_+(R)$ is infinitesimally cohesive, locally almost of finite presentation and nilcomplete.*

Lemma 12. *The functor $R \mapsto \iota\,\mathrm{Var}_+(R)$ has a connective cotangent complex.*

Proof. Let $X \in \mathrm{Var}_+(R)$. We need to study $\iota\,\mathrm{Var}_+(R \oplus M) \times_{\iota\,\mathrm{Var}_+(R)} \{X\}$. Using that $\iota\,\mathrm{Var}_+$ is infinitesimally cohesive we have a pullback diagram

$$\begin{array}{ccc} \iota\,\mathrm{Var}_+(R \oplus M) & \longrightarrow & \iota\,\mathrm{Var}_+(R) \\ \downarrow & & \downarrow \\ \iota\,\mathrm{Var}_+(R) & \longrightarrow & \iota\,\mathrm{Var}_+(R \oplus \Sigma M) \end{array}$$

so

$$\iota\,\mathrm{Var}_+(R \oplus M) \times_{\iota\,\mathrm{Var}_+(R)} \{X\} \simeq \iota\,\mathrm{Var}_+(R) \times_{\iota\,\mathrm{Var}_+(R \oplus \Sigma M)} \{X_{R \oplus \Sigma M}\}$$

This is the space of couples $(Y, \eta : Y_{R\oplus M} \xrightarrow{\sim} X_{R\oplus M})$ where Y is a variety over R. That is, identifying Y with X by the pullback of η with R, this is the space of maps $X_{R\oplus M} \to X_{R\oplus M}$ extending the identity over R. By the theory of the cotangent complex this is just $\mathrm{Map}_X(\mathbb{L}_{X/R}, \Sigma f^* M) \simeq \mathrm{Map}_R(\Sigma^{-1} f_+ \mathbb{L}_{X/R}, M)$. THerefore the cotangent complex at X is $\Sigma^{-1} f_+ \mathbb{L}_{X/R}$ □

Lemma 13. *For every simplicial set K with finitely many simplices in each dimension the functor $R \mapsto \iota\,\mathrm{Fun}(K, \mathrm{Var}_+(R))$ has a connective cotangent complex.*

Proof. We will prove more generally that for every map $K \to K'$ the map $\mathrm{Fun}(K', \mathrm{Var}_+) \to \mathrm{Fun}(K, \mathrm{Var}_+)$ has a connective cotangent complex. As usual one can reduce to the case $\partial\Delta^1 \to \Delta^1$. So we need to show that the diagonal map

$$\iota\,\mathrm{Fun}(\Delta^1, \mathrm{Var}_+(R)) \to \iota\,\mathrm{Var}_+(R) \times \iota\,\mathrm{Var}_+(R)$$

has a connective cotangent complex. To do so we need to describe the space of dotted arrows in the diagram

$$\begin{array}{ccc} \mathrm{Spec}\, R & \longrightarrow & \iota\,\mathrm{Fun}(\Delta^1, \mathrm{Var}_+(R)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R \oplus M) & \longrightarrow & \iota\,\mathrm{Var}_+ \end{array}$$

The bottom arrow corresponds to $X_{R\oplus M}, Y_{R\oplus M} \in \mathrm{Var}_+(R \oplus M)$ and the top arrow gives a map $f : X_R \to Y_R$ in $\mathrm{Var}_+(R)$. Therefore the space of dotted lifts is equivalent to the space of extensions to $X_{R\oplus M}$ of $X_R \to Y_R \to Y_{R\oplus M}$. The theory of cotangent complex tells us this is $\mathrm{Map}_X(f^*\mathbb{L}_{Y/R}, p^*M)$, that is $\mathrm{Map}_R(p_+f^*\mathbb{L}_{Y/R}, M)$. □

Lemma 14. *The functor sending R to the groupoid of abelian varieties is infinitesimally coehesive, locally almost of finite presentation, nilcomplete and admits a cotangent complex.*

Proof. That it is infinitesimally cohesive and nilcomplete follows from the fact that abelian group objects commute with limits. That it is locally almosto of finite presentation follows from the fact that abelian group objects commute with uniformly bounded above filtered colimits. So it is enough to look at the cotangent complex. But the trick here is to see that the map $\mathrm{Var}_+(R \oplus M) \to \mathrm{Var}_+(R)$ is conservative □