Spectral formal groups

Rmk: R $\mathbb{E}_\infty$-ring $\mathrm{Mod}_R^{b} \xrightarrow{\simeq} \mathrm{Mod}^{b}_{t\geq 0(R)}$

§1 Smooth coalgebras

Def: R $\mathbb{E}_\infty$-ring. The category of commutative (flat) R-coalgebras

$$\mathrm{cCAlg}_R^{(b)} := \mathrm{CAlg}\left(\left(\mathrm{Mod}_R^{(b)}\right)^{op}\right)^{op}$$

If R is discrete, then comm. R-coalg is just the data of

a) flat R-mod C

b) comult $\Delta_C : C \to C \otimes_R C$

c) counit : $\varepsilon_C : C \to R$

s.t. the diagrams

$$\begin{tikzcd}[column sep=small] & C \arrow[dl, "\Delta_C"'] \arrow[dr, "\Delta_C"] & \\ C \otimes_R C \arrow[rr, "\sigma"'] & & C \otimes_R C \end{tikzcd}$$

etc ..

commute.

<u>Ex:</u> R has trivial R-coalg structure.

<u>Ex:</u> R discrete & $M \in \mathrm{Mod}_R^\flat$.

The <u>divided powers coalg</u> is given by

$$\Gamma^*_R(M) := \bigoplus_{n \geq 0} (M^{\otimes n})^{\Sigma_n} \leftarrow \text{classical invariants!}$$

w/ obvious counit, comult given obvious maps

$$(M^{\otimes n+m})^{\Sigma_{n+m}} \longrightarrow (M^{\otimes n})^{\Sigma_n} \otimes (M^{\otimes m})^{\Sigma_m}$$

Def: R E_∞-ring. A coalgebra C is *smooth* if $\pi_0(C) \cong \Gamma^*_{\pi_0(R)}(M)$, where M is f.g. projective $/\pi_0(R)$

Grouplike elements

Def: C R-coalg. The space of *grouplike elements* is

$$GLike(C) := Map_{cCAlg_R}(R, C)$$

Note: if R is discrete, then

$$GLike(C) = \left\{ \eta \in C \;\middle|\; \begin{array}{l} \Delta_c(\eta) = \eta \otimes \eta \\ \varepsilon(\eta) = 1 \end{array} \right\}$$

$\exists$ universal property for $T^*_{\mathbb{R}}(M)$

§2. Étale descent

Smooth coalgebras / $\mathbb{R}$ $\mathbb{E}_\infty$-ring satisfy étale descent.

§3. Formal hyperplanes

<u>Def</u>: An <u>adic</u> $\mathbb{E}_\infty$-ring is a pair (A, τ)

A: $\mathbb{E}_\infty$-ring

τ: adic topology on $\pi_0(A)$.

A continuous map $A \to B$ of adic $\mathbb{E}_\infty$-algebras, is map $f: A \to B$ of $\mathbb{E}_\infty$-rings s.t. $\pi_0(f)$ is cont.

Rmk: $\exists$ a nice thy of I-adic completions for $\mathbb{E}_\infty$-rings, wher $I \subset \pi_0(R)$ an ideal.

Duality

M module / R $\mathbb{E}_\infty$-ring, denote
$M^\vee := \underline{Map}_R(M, R)$
The "duality" functor $Mod_R \circlearrowleft (\cdot)^\vee$ interchanges algebras & coalgebras

Ex: R discrete & M $\underbrace{\text{proj. f.g.}}$
then $T^*_R(M)^\vee \cong \overbrace{Sym^*_R(M^\vee)}$.
This $\overbrace{\text{has}}$ obvious adic topology
$Sym^{>0}_R(M^\vee)$

If R is $\mathbb{E}_\infty$, C smooth R-coalg, then $\pi_0(C^\vee) \cong \widehat{\mathrm{Sym}^*_R(M^\vee)}$

Point: $C^\vee$ is an adic comm. R-alg.

Def: If R is an adic $\mathbb{E}_\infty$-ring, then $\mathrm{Spf}(R)$ is the subfunctor of $\mathrm{Spec}(R)$ s.t.

discrete top on $\pi_0(A)$

$$\mathrm{Spf}(R)(A) := \{ R \to A \ \mathbb{E}_\infty\text{-map that is continuous} \}$$

Def: A functor $X : \mathrm{CAlg}^{cn}_R \to \mathcal{S}$ formal hyperplane if it is of the form $\mathrm{Spf}(C^\vee)$, where C is smooth coalg $/R$

Cospectrum of coalgebra

Two lemmas abt completion.

<u>Lem A</u> (Prop 1.3.13 ; EC II) C sm coalg $/R$ ($\uparrow$ $\mathbb{E}_\infty$)
$M \in \mathrm{Mod}_R$. Then

$$C^\vee \otimes_R M \longrightarrow \underline{\mathrm{Map}}_R(C, M)$$

is completion w.r.t. ideal

$$\mathrm{Sym}^{>0}_R(N^\vee) \subset \pi_0(C^\vee)$$

<u>Lem B</u> (Prop 1.3.15 ; EC II)
If C, D sm. coalg $/R$ ($\mathbb{E}_\infty$-ring), then

$$\mathrm{Map}_{c\mathrm{CAlg}}(D, C) \simeq \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, D^\vee)$$

Lets analyze $\mathrm{Spf}(C^\vee)$:

$$\mathrm{Spf}(C^\vee)(A) = \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, A)$$

$$\overset{\text{Lem A}}{\simeq} \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_A}((A \otimes_R C)^\vee, A)$$

$$\overset{\text{Lem B}}{\simeq} \mathrm{Map}_{\mathrm{cCAlg}_A}(A, A \otimes_R C)$$

$$= \mathrm{GLike}(A \otimes_R C)$$

Def: C sm. coalg $/R$, its cospectrum is

$$\mathrm{cSpec}(C)(A) := \mathrm{GLike}(A \otimes_R C)$$

Ex: Let $f: X \to \mathrm{Spec}(R)$ be
sp. alg. space

fiber-smooth (flat + π_0 smooth)

that admits a section $s: \mathrm{Spec}(R) \hookrightarrow X$.

Then $\hat{X}_s$ is a formal hyperplane:

étale locally $s: \mathrm{Spec}(R) \xhookrightarrow{V(x_1, \dots, x_r)} \mathrm{Spec}(R[x_1, \dots, x_r])$ "in"

& can compute the completion of .

§4. Spectral Formal gp

Def: A <u>spectral formal gp</u> $/R$

is a functor

$$G: \mathrm{CAlg}^{cn}_{\tau_{\geq 0}(R)} \to D_{\geq 0}(\mathbb{Z})$$

s.t. $\Omega^\infty \circ G : \mathrm{CAlg}^{cn}_{\tau_{\geq 0}(R)} \to \mathcal{S}$

is equiv to a formal hyperplane.

Ex: G strict comm fiber smooth alg gp $/R$ (e.g. $\mathbb{G}_m$, E str. ab. var)

Then the <u>formal completion</u> $\hat{G}$ is defined as

$$\underbrace{\hat{G}(A) \to G(A) \to G(A^{red})}$$

fiber sequence in $D_{\geq 0}(\mathbb{Z})$.

One checks that actually $\Omega^\infty \circ \hat{G}$ is described as the completion of G at zero section $\mathrm{Spec}(R) \xhookrightarrow{\{0\}} G$

§ 2. Étale descent etc.

everything discrete!

Prop: (Univ prop of $\Gamma^*_R(M)$)

$\theta: \mathrm{Hom}_{\mathrm{cCalg}_R}(C, \Gamma^*_R(M)) \hookrightarrow \mathrm{Hom}_R(C, M)$

$$M = \Gamma^1_R(M)$$

& $\mathrm{Im}(\theta)$ consists of maps f

(*) $\forall x \in C$

$$\boxed{f^{\otimes n}\left(\Delta^{(n)}_C(x)\right) = 0 \in M^{\otimes n} \quad \text{for } n \gg 0}$$

"Pf": One checks (cocomm. & coass.)

$$C \xrightarrow{\Delta^{(n)}_C} C^{\otimes n} \xrightarrow{f^{\otimes n}} M^{\otimes n}$$

factors thru $(M^{\otimes n})^{\Sigma_n}$. & under (*) these give a map $F: C \to \Gamma^*_R(M)$. ☐

Rmk: If $C = R$, M proj, then

$$\mathrm{GLike}(\Gamma^*_R(M)) = \{m \in M \mid m^{\otimes n} = 0 \text{ for } n \gg 0\}$$
$$= \mathrm{Map}_R(M^\vee, \sqrt{R})$$

$\Rightarrow$ If R reduced, C sm coalg $/R$, it has a unique gp like element.

Back to descent.

<u>Lem</u>: If $R \to R'$ is faithfully f and $C \in c\,\mathrm{CAlg}^b_R$ s.t.

1) C has a gplike elt

ii) $C' := R' \otimes_R C$ is smooth

$\Rightarrow$ C is smooth

<u>Def</u>: If $\eta \in C$ gplike, define

$\mathrm{Prim}_\eta(C)$ to be $\{x \in C \mid \Delta_C(x) = \eta \otimes x + x \otimes \eta$

<u>Note</u>: $1 \in \pi^\wedge_R(M)$ gplike &

$\mathrm{Prim}_1(\pi^*_R(M)) = \pi'_R(M)$

<u>Pf</u>: Let $C_0 := \pi^*_R(\mathrm{Prim}_\eta(C))$.

Let $\mathcal{F}$ be the sheaf (fflat top)

$$\mathcal{F}(A) := \left\{ \begin{array}{l} \text{isomorphisms } A \otimes_R C \xrightarrow{\psi} A \otimes_R C_0 \\ \text{s.t. } \psi(\eta) = 1 \in C_0 \\ \text{induces identity on primitive elements} \end{array} \right\}$$

Idea: want a global section of $\mathcal{F}(A)$.

<u>But</u>: $\mathcal{F}$ is a torsor / "quasi coherent sheaf" "$\square$"

+ another lem.

R reduced $\quad R'$ f.f. étale, $\quad R' \otimes_R R'$

$$\begin{array}{ccccccc} & & \exists \eta & \longmapsto & \eta' & \longmapsto & 0 \\ 0 & \to & C & \to & R' \otimes_R C & \rightrightarrows & R' \otimes_R R' \otimes_R C \\ & & & & \uparrow & & \uparrow \end{array}$$

$\exists!$ g plike elts

$\Rightarrow$ can use the above lemma

"$\square$"