Equivariant K-theory as tempered cohomology

§1) Equivariant K-theory

$\mathrm{Vect}_{\mathbb{C}}^{\simeq}$ = (ordinary) groupoid of finite dim. vect. spaces / $\mathbb{C}$.
(all morph. are isos)

For T a space, consider $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}) \simeq \mathrm{Fun}(\pi_{\leq 1} T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$
groupoid of functors (i.e. cat. of finite dim. complex local systems with invertible morphisms)

<u>Ex</u> if $T = BG$, (G finite group) Then $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}) \simeq$ category of finite dimensional G-representations

Classical topology on $\mathbb{C} \rightsquigarrow$ topological enrichment on $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$
Consider $N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$ – homotopy coherent nerve.

$\oplus$ of vector spaces $\rightsquigarrow$ symm. mon. str. on $\mathrm{Vect}_{\mathbb{C}}^{\simeq}$ and $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$
$\rightsquigarrow$ $\mathbb{E}_\infty$-structure on $N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$ (i.e. $N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$ is an $\mathbb{E}_\infty$-monoid in spaces).

<u>Def</u> $\underset{\text{connective spectrum.}}{\underbrace{ku(T)}} :=$ group completion of $N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$

(group completion = left adjoint $(\mathrm{CGroups} \xrightarrow{\text{forget}} \mathrm{CMonoids})$)

$\otimes$ of vector spaces $\rightsquigarrow$ symm. mon. str. on $\mathrm{Vect}_{\mathbb{C}}^{\simeq}$ and $\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$
which distributes over the first

$\Rightarrow N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})) \in \mathrm{CAlg}(\mathrm{CMon}(\mathcal{S}))$ — smash product of $\mathbb{E}_\infty$-spaces.
(i.e. an $\mathbb{E}_\infty$-semiring in spaces.)

$\Rightarrow ku(T) \in$ $\mathbb{E}_\infty$-rings in Spaces.

<u>Ex</u> 1) if T is contractible, $ku(T) \simeq ku$

2) $T = BG$
$\pi_0(N^{hc}(\mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))) = \{$ isom. classes of fin. dim. complex G-represent. $\}$

$\pi_0(ku(T)) \simeq \mathrm{Rep}(G)$

If $T \simeq BG$, $T' \simeq BG'$, $\forall f : T \to T' \rightsquigarrow f : \mathrm{Fun}(T', \mathrm{Vect}_{\mathbb{C}}^{\simeq}) \to \mathrm{Fun}(T, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$
compatible with $\oplus$, $\otimes$ and the topological enrichment
$\Rightarrow$ we get a map of $\mathbb{E}_\infty$-rings

$$f^* : ku(T') \longrightarrow ku(T).$$

If $G' = \{*\}$:
$$f^* : ku \longrightarrow ku(T)$$

i.e. <u>every $\mathbb{E}_\infty$-ring $ku(T)$ is a ku-algebra.</u>

<u>Rmk</u> G fin. group. $V_1, \dots, V_n$ s.t. $\{[V_1], \dots, [V_n]\} = \pi_0(\mathrm{Fun}(BG, \mathrm{Vect}_{\mathbb{C}}^{\simeq}))$

$$\left(\mathrm{Vect}_{\mathbb{C}}^{\simeq}\right)^n \xrightarrow{\ \simeq\ } \mathrm{Fun}(BG, \mathrm{Vect}_{\mathbb{C}}^{\simeq})$$
$$(W_1, \dots, W_n) \longmapsto \bigoplus_{i=1}^{n} V_i \otimes W_i$$

$\Rightarrow$ $ku(BG)$ and $KU(BG)$ are free of rank n over ku

<u>Notation</u> $KU(T) = ku(T) \otimes_{ku} KU \simeq ku(T)[\beta^{-1}]$

β = Bott element $\in \pi_2(ku)$

<u>Rmk</u> H finite abelian, $\widehat{H} = \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ Pontryagin dual, $\lambda \in \widehat{H}$

$$V_\lambda : H \to \mathbb{C}^\times \quad h \mapsto \exp(2\pi i \lambda(h))$$

$$\mathbb{Z}[\widehat{H}] \xrightarrow{\ \sim\ } \mathrm{Rep}(H)$$
$$\lambda \longmapsto [V_\lambda]$$

$\mathcal{T}^{op} \subset \mathcal{T}$ spaces of the form BH, H abelian finite.

Consider $ku : \mathcal{T}^{op} \to \mathrm{CAlg}_{ku}$

$$KU : \mathcal{T}^{op} \longrightarrow \mathrm{CAlg}_{KU}$$

Prop $ku : \mathbb{T}^{op} \longrightarrow \mathrm{CAlg}_{ku}$ is $\mathbb{P}$-divisible.

Recall: This means that the functor

$$\mathrm{Ab}_{fin} \xrightarrow{M \mapsto B\hat{M}} \mathbb{T}^{op} \xrightarrow{ku} \mathrm{CAlg}_{ku} \quad \text{is } \mathbb{P}\text{-divisible, i.e.}$$

1. it preserves finite coproducts ($M \oplus M' \mapsto ku(B\hat{M}) \otimes_{ku} ku(B\hat{M}')$)
2. $0 \to M' \to M \to M'' \to 0 \rightsquigarrow$

$$\begin{array}{ccc} ku(BM') & \longrightarrow & ku \\ \downarrow & & \downarrow \\ ku(BM) & \longrightarrow & ku(B\hat{M}'') \end{array}$$

finite flat of positive degree

Proof $ku(T)$ free / ku of finite rank $\Rightarrow$ flat

Fact $ku : \mathbb{T}^{op} \to \mathrm{CAlg}_{ku}$ is $\mathbb{P}$-div. iff.
- $ku(T)$ proj. ku-mod. of fin. rank
- $T \mapsto \pi_0(ku(T))$ determines a $\mathbb{P}$-divisible functor $\mathbb{T}^{op} \longrightarrow \mathrm{CAlg}_{\pi_0(ku)}$

Look at the functor

$$\begin{array}{ccc} \mathbb{T}^{op} & \longrightarrow & \mathrm{CAlg}_{\pi_0(ku)} \\ T & \longmapsto & \pi_0(ku(T)) \end{array}$$

By what we've said, this functor is $H \mapsto \mathrm{Rep}(H) \simeq \mathbb{Z}[\hat{H}]$
$\Rightarrow$ it coincides with the multiplicative $\mathbb{P}$-divisible groups $\mu_{\mathbb{P}^\infty}$?

The orientation of $\mu_{\mathbb{P}^\infty}$

$\mathbb{P}$-div. group associated to the $\mathbb{P}$-divisible functor $ku : \mathbb{T}^{op} \to \mathrm{CAlg}_{ku} \leftrightarrow (G, e)$ — G $\mathbb{P}$-divisible group, $e \in \mathrm{Pre}(G)$

Previous proof: $G_{\mathbb{Z}} = G \otimes_{ku} \pi_0(ku) \simeq \mu_{\mathbb{P}^\infty}$ of $\mathbb{P}$-div. groups over $\mathbb{Z}$

riguarda

$\mu_{\mathbb{P}^\infty}$ is Cartier dual to $\mathbb{Q}/\mathbb{Z}$ $\Rightarrow$ has no nontrivial deform.

étale $\mathbb{P}$-div. group.

riguarda

$\Rightarrow \exists$ essent. unique lift $G \simeq \mu_{\mathbb{P}^\infty}$ of $\mathbb{P}$-div. groups over ku

$\Rightarrow e \in \mathrm{Pre}(G) \rightsquigarrow B(\mathbb{Q}/\mathbb{Z}) \longrightarrow GL_1(ku)$

$\longleftrightarrow \Sigma^\infty_+ B(\mathbb{Q}/\mathbb{Z}) \longrightarrow ku$ (map of E_∞-rings)

$$\Sigma^\infty_+ B(\mathbb{Q}/\mathbb{Z}) \longrightarrow \underset{\substack{\parallel \\ \Sigma^\infty_+ K(\mathbb{Z},2)}}{\Sigma^\infty_+ \mathbb{CP}^\infty} \xrightarrow{\rho} ku$$

riguardo: $\mathbb{CP}^\infty \simeq BU(1) \subset N^{hc}(\mathrm{Vect}_{\mathbb{C}}^{\simeq}) \to \Omega^\infty(ku)$ carrying can. gen. of $\pi_2(\mathbb{CP}^\infty)$ to $\beta \in \pi_2(ku)$

Cor $KU : \mathcal{T}^{op} \longrightarrow \mathrm{CAlg}_{KU}$ is $\mathbb{P}$-divisible

It correspond to the $\mathbb{P}$-div. group $\mu_{\mathbb{P}^\infty}$ over KU equipped with the orientation described above.

Tempered cohomology

$A \in \mathrm{CAlg}$, $G \in \underbrace{BT(A)}_{\substack{\infty\text{-cat} \\ \text{of } \mathbb{P}\text{-div.} \\ \text{groups}/A}} + e \in \mathrm{Pre}(G)$

$$A_G : \mathcal{T}^{op} \to \mathrm{CAlg}_A \rightsquigarrow (G, e)$$

Notation $\quad A_G(T) =: A_G^T$

If $H \in \mathrm{Ab}_{fin}$, $\mathrm{Spec}(A_G^{BH}) \overset{\text{canonical}}{=} G[\hat{H}]$ $\quad (\hat{H} = \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z}))$

$\underset{\text{non canonical}}{\text{IS}}$

$G(\pi_1(T))$

Construction (Tempered function spectra) $A \in \mathrm{CAlg}$, (G, e)

Identify $\mathcal{T}$ with ess. image of $\mathcal{T} \hookrightarrow \mathcal{OS}$ $\quad (T \mapsto T^{(-)})$

$$\begin{array}{ccc} A_G : \mathcal{T}^{op} & \longrightarrow & \mathrm{CAlg}_A \\ \downarrow & \nearrow_{A_G} & \\ \mathcal{OS}^{op} & & \end{array}$$

$A_G \leftarrow$ ess. unique ext. which preserves small limits

$\mathbb{X} \in \mathcal{OS}$; $A_G^{\mathbb{X}} := A_G(\mathbb{X})$ G-tempered function spectrum

NOTATION if $\mathbb{X} = X^{(-)}$, $\rightsquigarrow$ A_G^{X}

$A_G^{\mathbb{X}}$ is essentially determined as a spectrum by the formula

$$\Omega^{\infty-n}(A_G^{\mathbb{X}}) \simeq \mathrm{Map}_{\mathcal{OS}}(\mathbb{X}, \Omega^{\infty-n} A_G)$$

Construction (Tempered cohomology)

$$A_G^*(\mathbb{X}) := \pi_{-*}(A_G^{\mathbb{X}})$$ G-tempered cohomology ring of $\mathbb{X}$

Equivariant K-theory as tempered cohomology

$\mu_{\mathbb{P}^\infty}$:= multiplicative $\mathbb{P}$-div. group over KU with its orientation.

$$\mathbb{X} \mapsto KU^{\mathbb{X}}_{\mu_{\mathbb{P}^\infty}} \quad , \quad \mathbb{X} \mapsto KU^*_{\mu_{\mathbb{P}^\infty}}(\mathbb{X})$$

$\mu_{\mathbb{P}^\infty}$ - tempered function spectrum $\mu_{\mathbb{P}^\infty}$ - tempered cohomology ring

COR $\Rightarrow$ $KU^{T}_{\mu_{\mathbb{P}^\infty}} \simeq KU(T)$ functorial in $T \in \mathcal{T}$

G finite abelian group, $KU^0_{\mu_{\mathbb{P}^\infty}}(BG) \simeq \mathrm{Rep}(G)$

Aim extend this to all finite groups

more generally, for any G-space X:

$$KU^*_{\mu_{\mathbb{P}^\infty}}(X//G) \simeq KU^*_G(X) \quad \text{canonical.}$$

Main features of G-equivariant K-theory

∞-cat. of G-spaces

(a) $X \mapsto KU_G^X : \mathcal{S}_G^{op} \longrightarrow \mathrm{CAlg}_{KU}$ which carries small colimits of $\mathcal{S}_G$ to small limits in CAlg_{KU}

(b) $\mathrm{Orbit}(G) \subset \mathcal{S}_G$

$$\mathrm{Orbit}(G)^{op} \hookrightarrow \mathcal{S}_G^{op} \xrightarrow{X \mapsto KU_G^X} \mathrm{CAlg}_{KU}$$
$$X \longmapsto KU(\underbrace{X_{hG}}_{\text{homotopy orbit space}})$$

If $X = H\backslash G$ $(H \subseteq G)$

$$KU_G^X \simeq KU(BH), \quad KU_G^0(X) \simeq \mathrm{Rep}(H)$$

(c) $X \in \mathrm{Top}$, $G \curvearrowright X$, $X = \mathrm{Sing}_\bullet^G(X) \in \mathcal{S}_G$

$$\{\underbrace{G\text{-equiv. complex vect. bundles on } X}_{\text{isom.}}\} \xrightarrow[\text{canonical}]{\mathcal{E} \mapsto [\mathcal{E}]} KU_G^0(X)$$

If X finite G-space wi orbits $KU_G^0(X)$ as the Grothendieck group of the monoid on the left.

(d) $G \curvearrowright X$, $\mathcal{E}$ G-equiv. vector bundle of rank n on X

$Y = \mathbb{P}(\mathcal{E})$; $\pi : Y \to X$

tautological s.e.s. $\quad 0 \to \underbrace{\mathcal{O}(-1)}_{\text{rank } 1} \to \pi^*\mathcal{E} \to \mathcal{Q} \to 0$

$\forall d \in \mathbb{Z} \quad \mathcal{O}(d) = \left(\mathcal{O}(-1)^\vee\right)^{\otimes d}$

$X = \mathrm{Sing}_\bullet^G(X)$, $Y = \mathrm{Sing}_\bullet^G(Y)$

$$\{[\mathcal{O}(d)]_{0 \le d < n}\} \text{ basis of } KU_G^*(Y) \text{ over } KU_G^*(X)$$

Rmk $X \mapsto KU_G^X : \mathcal{J}_G^{op} \longrightarrow CAlg_{KU}$ is char. by (a) and (b) above

(i.e. right Kan ext. of $\mathrm{Orbit}(G)^{op} \longrightarrow CAlg_{KU}$

along $\mathrm{Orbit}(G)^{op} \subset \mathcal{J}_G^{op}$ $\quad X \longmapsto KU(X_{hG})$

THM G finite group, $X \in \mathcal{J}_G$. There is a canonical equiv.

$$KU_G^X \simeq KU_{\mu_{p^\infty}}^{X/\!/G}$$

In particular, $\exists$ canonical isom. of graded rings

$$\underset{\text{K theory}}{\text{G equiv.}} \rightarrow KU_G^*(X) \simeq KU^*_{\mu_{p^\infty}}(X/\!/G) \leftarrow \text{tempered cohomology}$$

Ex $X \simeq *$, $\quad KU^0_{\mu_{p^\infty}}(BG) \simeq \mathrm{Rep}(G)$

Proof $\mathrm{Orbit}(G)_{ab} := \{ H\backslash G \in \mathrm{Orbit}(G) : H \le G \text{ abelian} \}$

$$\subseteq \mathrm{Orbit}(G)$$

$\mathrm{Orbit}(G)_{ab} \subseteq \mathrm{Orbit}(G) \subseteq \mathcal{J}_G$, if $X \in \mathrm{Orbit}(G)_{ab}$,

$\overset{(b)}{\Longrightarrow}$ $\quad u_X : KU_G^X \simeq KU(X_{hG}) \simeq KU_{\mu_{p^\infty}}^{X/\!/G}$ $\quad X_{hG} \in \mathcal{J}$ canon. equiv.

$\mathcal{J}_G^{op} \longrightarrow CAlg_{KU}$ $\quad X \mapsto KU_{\mu_{p^\infty}}^{X/\!/G}$ right Kan extension of its restriction to $\mathrm{Orbit}(G)_{ab}$

$\Rightarrow \underset{X}{\overset{\mathrm{Orbit}(G)_{ab}}{\Psi}} \longmapsto u_G$ extends to nat. transf.

$$KU_G^X \longrightarrow KU_{\mu_{p^\infty}}^{X/\!/G}$$

$X \mapsto \mathcal{U}_X$ carries colimits in $\mathcal{T}_G$ to limits in ~~CAlg KU~~

$\Rightarrow$ it suffices to show that $\mathcal{U}_X$ is an equiv. when X is a G-orbit.

If this is not the case, $\exists$ $H \subset G$ such that $\mathcal{U}_{H\backslash G}$ is not an equivalence.

Choose H such that $|H|$ is minimal.

H can't be abelian $\Rightarrow$ it admits an irred. representation V of dimension $\gneq 1$.

$$V \longleftrightarrow G\text{-equiv. vector bundle } \mathcal{E} \text{ on } H\backslash G = X$$

$$Y_0 = \mathbb{P}(\overset{\curvearrowleft G}{\mathcal{E}}) = \mathrm{Sing}^G_\bullet(Y_0) \in \mathcal{T}_G$$

$$Y_\bullet = \check{\mathrm{C}}\text{ech nerve of } Y_0 \to X$$

$$\begin{array}{ccc} KU_G^X & \xrightarrow{\mathcal{U}_X} & KU_{\mu_{p^\infty}}(X/\!\!/G) \\ v\downarrow & & \downarrow w \\ \mathrm{Tot}(KU_G^{Y_\bullet}) & \xrightarrow[\mathrm{Tot}(\mathcal{U}_{Y_\bullet})]{} & \mathrm{Tot}(KU_{\mu_{p^\infty}}^{Y_\bullet/\!\!/G}) \end{array}$$

- $KU_G^{Y_0}$ is a faith. flat KU_G^X-algebra and $KU_G^{Y_\bullet}$ is the cosimplicial KU_G^X-algebra given by iterated $\otimes$-powers of $KU_G^{Y_0}$
  $\Rightarrow$ v is an equiv. by faith. flat descent.
- $Y_\bullet/\!\!/G$ = Čech nerve of $Y_0/\!\!/G \xrightarrow{\pi} X/\!\!/G$
  It suffices to show that π is an effective epi of orbispaces
  $\Leftrightarrow$ $\forall$ abelian subgroup $A \subseteq G$, $\forall x \in X$ fixed by A
  $\exists$ $y \in Y_0$ lying over x which is fixed by A

WLOG $\quad x =$ Identity coset $\in H\backslash G$

$\Rightarrow$ A is an abelian subgroup of H.

$\Rightarrow$ $y \in Y_0^A \longleftrightarrow$ 1-dim. complex subspace $L \subseteq V$ fixed by A

But A abelian $\Rightarrow$ $A \curvearrowright V$ decomposes as direct sum of 1-dim. representations.

- we will show that every map u_{Y_k} $(k \geq 0)$ is an equiv.

Write Y_k as colimit of G-orbits $H' \backslash G$

$\Rightarrow$ we are reduced to show that $u_{H' \backslash G}$ is an equiv. whenever $\exists\ y \in Y_k^{H'}$; x = image of y in $X = H \backslash G$

Replacing H' by a conjugate subgroup, wLOG x = identity coset $\in H \backslash G$, so that $H' \leq H$

$\overset{\text{minimality}}{\Longrightarrow}$ $H' = H$

assumpt.

$\exists$ of a fixed point $y \in Y_k^H$ lying over $x \in H \backslash G$ identity coset implies that V contains a 1-dim. complex subspace $L \subseteq V$ fixed by H

As V is irreducible, $L = V$ $\lightning$ $(\dim(V) > 1)$

$\Rightarrow$ v, w, $\mathrm{Tot}(u_{Y_\bullet})$ are equiv.

$\Rightarrow$ u_X is an equiv. $\lightning$

$\square$