REGENSBURG RESEARCH SEMINAR SS 2023

SELMER K-THEORY

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Selmer K-theory is a localizing invariant of stable categories introduced by Clausen to give a K-theoretic construction of the Artin map from the idele class group of a number field to its abelianized Galois group. For schemes, Selmer K-theory is closely related to the étale sheafification of algebraic K-theory, and in general it can thus be viewed as a non-commutative extension of the latter.

In this seminar, we will review the definition of Selmer K-theory, which combines insights of Thomason on K(1)-local algebraic K-theory and of Geisser–Hesselholt on topological cyclic homology. We will then discuss applications to étale K-theory following [CM21].

1. **Introduction (18.04)**
2. **Cyclotomic spectra and topological Hochschild homology (25.04)**
   Give the definition of (bounded below) cyclotomic spectra and p-cyclotomic spectra after Nikolaus–Scholze [NS18]. Explain the definition of $\mathrm{THH}(\mathcal{C})$ for a stable ∞-category $\mathcal{C}$ and the construction of the cyclotomic structure on $\mathrm{THH}(\mathcal{C})$. Define $\mathrm{TC}^{\pm}(\mathcal{C})$, $\mathrm{TP}(\mathcal{C})$, and $\mathrm{TC}(\mathcal{C})$,
3. **Finiteness properties of TC (02.05)**
   [CMM21, Section 2]. Review the definition of $\mathrm{TR}^*(\mathcal{C})$ with the operators R, F, V and its relationship to $\mathrm{TC}(\mathcal{C})$ (aka the "classical" definition of TC). Show that the functor $\mathrm{Map}(\mathbf{1}, -)/p\colon \mathrm{CycSp}_{\geq 0} \to \mathrm{Sp}$ preserves colimits [CMM21, Theorem 2.7], hence that $\mathrm{TC}/p\colon \mathrm{Alg}(\mathrm{Sp}_{\geq 0}) \to \mathrm{Sp}$ preserves sifted colimits [CMM21, Corollary 2.15].
4. **The cyclotomic trace (09.05)**
   Explain the definition of noncommutative motives and the representability of K-theory therein [BGT13]. Prove that THH is a localizing invariant of stable ∞-categories ([BM12] or [HSS17, Section 3, 4.5]), which gives rise to the cyclotomic trace $\mathrm{K} \to \mathrm{TC}$. State the Dundas–Goodwillie–McCarthy theorem, that the fiber $\mathrm{K}^{\mathrm{inv}}$ is nil-invariant on $\mathrm{Alg}(\mathrm{Sp}_{\geq 0})$.
5. **KU- and K(1)-localization (16.05)**
   Define E-localization $L_E\colon \mathrm{Sp} \to \mathrm{Sp}$ for a spectrum E. Show that L_{KU} is a smashing localization and that $L_{\mathrm{KU}}(X) \otimes \mathbb{Q} = X \otimes \mathbb{Q}$ [Bou79]. Define also the first Morava K-theory K(1) at a prime p so that $L_{\mathrm{K}(1)}(X) = L_{\mathrm{KU}}(X)^{\wedge}_p$. Explain the telescope conjecture (at height 1), which describes $L_{\mathrm{K}(1)}$ as a telescopic/finite localization. Define Selmer K-theory $\mathrm{K}^{\mathrm{Sel}}(\mathcal{C})$ [Cla17, Definition 0.4].
6. **The Geisser–Levine and Geisser–Hesselholt theorems (23.05)**
   Let R be an ind-smooth local $\mathbb{F}_p$-algebra. The Geisser–Levine theorem [GL00] identifies the homotopy groups $\pi_n(\mathrm{K}(R)/p)$ with the logarithmic de Rham groups $\nu^n(R) = \ker(1 - C^{-1}) \subset \Omega^n_R$. The Geisser–Hesselholt theorem [GH99] shows that $\pi_n(\mathrm{TC}(R)/p)$ is an extension of $\nu^n(R)$ by $\tilde{\nu}^{n+1}(R) = \mathrm{coker}(1 - C^{-1})$. Explain the statements and give an overview of the proofs (see also [CMM21, Theorem 4.29]).

7. **The K-theory of henselian pairs (06.06)**
[CMM21, Section 4]. Sketch the proof of the main result of [CMM21], that the fiber of the cyclotomic trace $\mathrm{K}_{\geq 0} \to \mathrm{TC}$ satisfies rigidity for all henselian pairs (A, I).
8. **Preliminaries on hypersheaves (13.06)**
[CM21, Section 2]. Recall the notions of hypercompleteness and Postnikov completeness for prestable ∞-categories, and prove that they agree when the cohomological dimension is bounded [CM21, Proposition 2.10, 2.19]. Discuss the phenomenon of smashing hypercompletion and the criterion [CM21, Proposition 2.28, 2.31]. Define (weak) nilpotence and (weak) rapid convergence and prove the hypercompleteness criterion [CM21, Proposition 2.35].
9. **The Nisnevich topos (20.06)**
[CM21, Section 3]. Prove that locally finite homotopy dimension implies Postnikov completeness of objects [Lur03, Proposition 4.1.5]. Prove [CM21, Corollary 3.12]. Review the Nisnevich topology on qcqs spectral algebraic spaces and prove that the homotopy dimension of the Nisnevich topos is bounded by the Krull dimension [CM21, Theorem 3.18].
10. **The étale topos (27.06)**
[CM21, Section 4]. Define the classifying ∞-topos of a profinite group as a sheaf topos. Describe sheafification and hypersheafifcation [CM21, Proposition 4.9, 4.12] and deduce the hypercompleteness criterion and smashing property [CM21, Proposition 4.16, 4.17]. Prove "étale = Nisnevich + Galois" [Lur18, B.6.4.1] and the main hypercompleteness criterion for étale sheaves [CM21, Theorem 4.38].
11. **Étale descent for TC and KU-localized invariants (04.07)**
Prove Theorem 5.16 and Corollary 5.39 (in the case $n = 1$) in [CM21].
12. **Selmer K-theory and étale K-theory (11.07)**
Prove that $\mathrm{K}^{\mathrm{Sel}} \colon \mathrm{CAlg}(\mathrm{Sp}_{\geq 0}) \to \mathrm{Sp}$ is a finitary étale sheaf [CM21, Theorem 6.6], and that the induced map $\mathrm{K}^{\text{ét}} \to \mathrm{K}^{\mathrm{Sel}}$ is an isomorphism in degrees ≥ -1 [CM21, Theorem 7.12]. Finally, show that $\mathrm{K}^{\mathrm{Sel}}_{(p)}$ is an étale hypersheaf on schemes of finite Krull dimension and bounded p-local cohomological dimension [CM21, Corollary 7.15].

References

[BGT13] A. J. Blumberg, D. Gepner, and G. Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, pp. 733–838, preprint arXiv:1001.2282

[BM12] A. J. Blumberg and M. A. Mandell, *Localization theorems in topological Hochschild homology and topological cyclic homology*, Geom. Topol. **16** (2012), preprint arXiv:0802.3938

[Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, pp. 257–281

[Cla17] D. Clausen, *A K-theoretic approach to Artin maps*, 2017, arXiv:1703.07842

[CM21] D. Clausen and A. Mathew, *Hyperdescent and étale K-theory*, Invent. Math. **225** (2021), pp. 981–1076, preprint arXiv:1905.06611

[CMM21] D. Clausen, A. Mathew, and M. Morrow, *K-theory and topological cyclic homology of henselian pairs*, J. Amer. Math. Soc. **34** (2021), no. 2, pp. 411–473, preprint arXiv:1803.10897

[GH99] T. Geisser and L. Hesselholt, *The topological cyclic homology of schemes*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., 1999, pp. 41–87

[GL00] T. Geisser and M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), no. 3, pp. 459–493

[HSS17] M. Hoyois, S. Scherotzke, and N. Sibilla, *Higher traces, noncommutative motives, and the categorified Chern character*, Adv. Math. **309** (2017), preprint arXiv:1511.03589

[Lur03] J. Lurie, *On ∞-topoi*, 2003, arXiv:0306109

[Lur18] J. Lurie, *Spectral Algebraic Geometry*, 2018

[NS18] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, pp. 203–409, preprint arXiv:1707.01799