KU- AND K(1)-LOCALIZATION

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Abstract. In this note, we give a brief overview of E-localization and explain telescope conjecture (at height 1), which describes $L_{K(1)}$ as a telescopic/finite localization. In particular we show that L_{KU} is a smashing localization, define Morava K-theory $K(1)$ at a prime p and show that $L_{K(1)}(X) = L_{KU}(X)^{\wedge}_p$ and eventually define Selmer K-theory $K^{\mathrm{Sel}}(\mathcal{C})$.

Contents

This document contains slightly expanded version of an overview talk, delivered at Selmer K-theory Oberseminar organized by Prof. Marc Hoyois, on the topic KU- and K(1)-localization [Cla17]. these notes are entirely expository and are not aimed to give a comprehensive account; rather, we hope it might be seen as a bridge between chromatic homotopy theory and algebraic K-theory.

We have augmented the original content of the talk by some materials well known to the audience of this subject. In particular, we have proved some results that were recalled during the talk. We will assume some familiarity with basic notions from stable homotopy theory, chromatic homotopy theory and algebraic K-theory, the interested reader can refer to [Lur17], [Rav92a] and [TT90] respectively for the same.

1. E-localization

We will start with some historical context to get motivation for introducing E-localization. The stable homotopy category $\mathcal{S}\mathrm{p}$ is extraordinarily complicated. However, a set of approximations and localizations to it that are much simpler and closer to algebra. The stable homotopy category is somewhat analogous to the derived category of a ring R, only that R is replaced by the stable sphere $\mathbb{S}$. For simplicity, we always consider the p-local stable homotopy category and the p-local sphere for some prime p. It is very common to study R-modules using the fields over R, and in recent years there has been an enormous work on studying the stable homotopy category via its fields. These fields are referred to as Morava K-theories, denoted by $K(n)$ introduced by Morava in early 1970's.

Associated to Morava K-theories are various (homotopy) categories of local spectra that are the approximations to the stable homotopy category mentioned above. There is the category $\mathcal{L}$ of spectra local with respect to $K(0) \vee \cdots \vee K(n)$ and the category $\mathcal{K}$ of spectra local with respect to $K(n)$. There categories are themselves stable homotopy categories, in the sense of Hopkins [HS98]. We will show that the category $\mathcal{K}$ is in a certain sense irreducible; that is, it has no nontrivial further localizations.

We will speak about a number of results from Chromatic homotopy theory (such as the smashing theorem, telescopic conjecture) which provides an understanding of $\mathcal{K}$ for all n and p will give complete information about $\mathcal{S}$.

Definitions 1.1. Let $E \in \mathcal{S}\mathrm{p}$ be a spectrum.

- A spectrum X is called E-*acyclic* if $E \otimes X \simeq 0$. And we denote $\mathrm{Null}_E = \{X \in \mathcal{S}\mathrm{p} | E \otimes X \simeq 0\}$
- X is called E-*local* if any map from an E-acyclic spectrum Y into X is nullhomotopic (i.e., $Y \in \mathrm{Null}_E \Rightarrow [Y, X] \simeq 0$).
- $f : X \to Y$ is called E-*equivalence* if $E \otimes f : E \otimes X \to E \otimes Y$ is an equivalence or, equivalently, if the fiber of f is E -acyclic in other words, a map is an E-equivalence if and only if it induces an equivalence in E-homology.

Notation 1.2. We will denote the full subcategory of E-local spectra by $\mathcal{S}\mathrm{p}_E$.

A localization functor is an endofunctor L of the stable homotopy category together with a natural transformation $\eta : \mathrm{id} \to L$ such that $L\eta : L \to L^2$ is an equivalence and $L\eta \simeq \eta L$. These localization functors were first discussed in Adams' blue book [Ada74], Bousfield in his [Bou79] came up with a non-computational existence proof for such localization functor which forces the E-equivalence to be invertible.

Theorem 1.3. *If E is a spectrum, then there exists a functor*

$$L_E : \mathcal{S}p \to \mathcal{S}p$$

and a natural transformation $\eta_E : id \to L_E$ such that for any spectrum X,

(a) *the map $\eta_E(X) : X \to L_E X$ shows that $L_E X$ is E-local.*
(b) *the map $X \to L_E X$ is an E-equivalence : that is, it induces an isomorphism on E-homology groups $E_*(X) \simeq E_* L_E(X)$.*

The functor L_E is called Bousfield localization at E and the fiber M_E of η_E is called E-acyclization.

So the fiber $M_E X \to X \to L_E X$ is E-acyclic, so if a spectrum Y is E-local then the restriction map $[L_E X, Y] \to [X, Y]$ is an isomorphism, and $\mathrm{Map}(L_E X, Y) \to \mathrm{Map}(X, Y)$ is a weak equivalence. So L_E can be regarded as a left adjoint to the inclusion $\mathcal{S}\mathrm{p}_E \subset \mathcal{S}\mathrm{p}$ of the full subcategory of E-local spectra.

We need prove the existence of this localization functor, Adams tried to do this by directly localizing the homotopy category, but this procedure is more rigorous and combinatorial. The hint is, a localization of a locally small category need not be locally small. Bousfield in [Bou79] uses this hint and some clever idea from cardinalities of their spectra, a trick later called as "Bousfield-Smith cardinality argument" [Hir03][§4.5]. The proof stretches until Theorem 1.8.

Recall that a subspectrum B of a CW-spectrum X is closed if B is a union of cells and any cell of X with some suspension in B is in B; this assures that X/B is a CW-spectrum.

Lemma 1.4. *Let X be a CW-spectrum and B a proper closed subspectrum with $E_*(X,B)=0$, and let us assume κ be an infinite cardinal $\geq |\pi_* E|$. Then there is a closed subspectrum $W \subseteq X$ with at most κ cells that does not contain in B and $E_*(W, W\cap B)=0$.*

Proof. It's enough to prove that the collection of spectra W with $|E_*W| \leq \kappa$ is closed under κ-small colimits. Let W_1 be any closed subspectrum of X not contained in B and with at most κ cells. Inductively, for any W_n and for any class $\alpha \in E_*(W_n, W_n \cap B)$, choose a finite closed subspectrum $V_\alpha \in X$ such that α goes to zero in $E_*(W_n \cup V_\alpha, (W_n \cup V_\alpha)\cap B)$, and let W_{n+1} be the union of all W_n along with all V_α. By assumption, if W_n has at most κ-cells, then $E_*(W_n)$ has at most κ elements since $\kappa \geq |\pi_* E|$; thus by induction, all W_n have at most κ-cells. And constructing $W = \operatorname{colim} W_n$; it is clear that $E_*(W, W\cap B)=0$; that is W is not contained in B, and that W has at most κ cells. □

Lemma 1.5. *For any E, there exists an E-acyclic spectrum A such that a spectrum X is E-local if and only if $[A,X]\simeq 0$*

Proof. Choosing κ as before, and let $\{K_\alpha\}$ be a set of weak equivalence classes of E-acyclic spectra with at most κ cells. Let $A = \bigvee_\alpha K_\alpha$. Clearly if X is E-local, then $[A,X]\simeq 0$ (from definition). Conversely, if $[A,X]\simeq 0$, then $[A',X]\simeq 0$ for any spectrum A' that is obtained from A by taking weak equivalences, shifts, wedges, summands, and cofibers. Let $C(A)$ denote this class of spectra, so now it suffices to prove that every E-acyclic spectrum of A belongs in $C(A)$.

Let X be a E-acyclic spectrum; up to weak equivalence, we can consider X to be a CW-spectrum. By the previous lemma we can construct,

$$0 = B_0 \subset B_1 \subseteq \cdots B_n = X$$

such that

(i) each B_r is an E-acyclic subspectrum,
(ii) each B_{r+1} is obtained from B_r by adding W_r as in the previous lemma,
(iii) for r a limit ordinal, $B_r = \bigcup_{k<r} B_k$.

Now, if $B_r \in C(A)$, there is a cofiber sequence

$$B_r \to B_{r+1} \to K_\alpha,$$

where K_α is weakly equivalent to E-acyclic spectrum $W_r/(W_r \cap B_r)$, and thus a cofiber sequence

$$\Omega K_\alpha \to B_r \to B_{r+1},$$

thus B_{r+1} is also in $C(A)$. Likewise, if r is a limit ordinal, it is the cofiber of

$$\bigvee_{k<r} B_k \xrightarrow{1-i} \bigvee_{k+1<r} B_{k+1} \to B_r,$$

here i is the wedge of $B_k \hookrightarrow B_{k+1}$. By transfinite induction, all B_r, and in particular X belongs in $C(A)$. □

A lot happened in the field of categories since Bousfield's paper, so we will prove a lot of things in more mordern language but still stick to the spirit of the original paper. Before we prove the final part of the existence a localization functor, we will quickly look at the small object argument which plays a key role in the proof. For a reference on the small object argument, check [Hov99][§2.1.2]

1.6. We will briefly discuss how, Bousfield localizations can be formed by the small object argument with the colimits in our category.

A known argument from Kan is that we can replace the mapping space criterion for local objects with a lifting criterion when $\mathcal{C}$ has homotopy colimits using the notions of ∞-categories.

Given a map $f_0 : A_0 \to B \in \mathcal{C}$, we can construct a double mapping cylinders $f_n : A_n \to B$ and we find that an object Y is f_0-local if and only if ever map $g : A_n \to Y$ can be extended to a map $\overline{g} : B \to Y$ up to homotopy. More generally, If S and T are classes of morphisms and a collection of maps S is containing in a larger T closed under double mapping cylinders, and ask whether Y satisfies an extension property with respect to T.

Inductively, first start with $Y_0 = Y$. For a given Y_k, either Y_k is local (in this case, we are done) or there exists some set of maps $A_i \to B_i$ in T and the maps $g_i : A_i \to Y_n$ which do not extend B_i up to homotopy. The homotopy pushout

$$\bigsqcup_i B_i \leftarrow \bigsqcup_i A_i \to Y_k$$

and call this as Y_{k+1}. The map $Y_k \to Y_{k+1}$ is an S-equivalence because it is a homotopy pushout along an S-equivalence, and all the solutions for the extension problem in Y_k now contain in Y_{k+1}.

So following the above argument we construct $Y_0, Y_1, Y_2, \ldots$, and define $Y_s = \operatorname{ho\,colim} Y_n$. Once we have constructed Y_k for all the ordinals $k < r$, we define a new $Y_r = \operatorname{ho\,colim} Y_k$. The map $Y \to Y_r$ is a homotopy colimit of S-equivalence and hence S-equivalence.

Now, the problem is we need to stop this procedure at some point, like some ordinal r which is extremely large such that any map $A_i \to Y_r$ factors, up to homotopy, through some object Y_k with $k < r$. This could happen due to the compactness property of the object A_i, and this is called the *small object argument*. On the point-set level this has been studied by Smith's theory of combinatorial model categories and on the homotopical level this can be addressed using Lurie's theory of presentable ∞-categories.

Remark 1.7. If our category $\mathcal{C}$ does not have enough colimits, the small object argument may not apply. But, this doesn't stop Bousfield localizations from existing as they don't depend upon this particular construction in the argument.

Theorem 1.8. *For any E, there exists a localization functor $X \mapsto [X \to L_E X]$.*

Proof. From the previous lemma, all we need is a canonical map $X \to L_E X$ such that $[A, L_E X] \simeq 0$. Getting ideas from the small object argument, we can do this by successively coning off all maps from A and using transfinite induction. By initial construction A is a wedge of spectra with less than κ cells, each of which should be κ-small, so A is κ-small (could be even bigger than the one we chose earlier) and the small object argument goes through. This also gives the functoriality of $A \to L_E X$. □

Example 1.9. Lets consider E be the Eilenberg-McLane spectrum $H\mathbb{Q}$. Then a spectrum X is E-acyclic if and only if the homotopy groups $\pi_* X$ consist entirely of torsion. A spectrum X is E-local if and only if the homotopy groups $\pi_* X$ are rational vector spaces.

Example 1.10. Bousfield localization can also be found in chain complexes of abelian groups as well. Fixing a prime p, we say that a projective chain complex $X_\bullet$ is $\mathbb{Z}/p\mathbb{Z}$-acyclic if $X_\bullet \otimes \mathbb{Z}/p\mathbb{Z}$ is nullhomotopic: equivalently, $X_\bullet$ is $\mathbb{Z}/p\mathbb{Z}$-acyclic if each homotopy group $H_n(X_\bullet)$ is a $\mathbb{Z}[\frac{1}{p}]$-module. We say that $X_\bullet$ is $\mathbb{Z}/p\mathbb{Z}$-local if every map from a projective $\mathbb{Z}/p\mathbb{Z}$-acyclic chain complex into $X_\bullet$ is nullhomotopic.

For any projective chain complex $X_\bullet$, we can define its completion $\widehat{X_\bullet}$ to be the homotopy limit

$$\varprojlim_n X_\bullet \otimes \mathbb{Z}/p\mathbb{Z}.$$

As a homotopy limit of $\mathbb{Z}/p\mathbb{Z}$-local chain complexes, we conclude that $\widehat{X_\bullet}$ is $\mathbb{Z}/p\mathbb{Z}$-local. On the other hand, we can show that $X_\bullet \to \widehat{X_\bullet}$ induces a quasi-isomorphism module p, so that $\widehat{X_\bullet}$ can be identified with the $\mathbb{Z}/p\mathbb{Z}$-localizations of $X_\bullet$.

1.11. It is good to think Bousfield localization as involving a mix of above examples. Something like, it behaves as a restriction to an open scheme and sometimes as a completion along a closed subscheme.

1.12. (**Bousfield equivalence**) id $\to L_E$ depends only on Null_E. Two spectra E and F are *Bousfield equivalent* if $\mathrm{Null}_E = \mathrm{Null}_F$. The Bousfield class $< E >$ of E is the equivalence class of E under the following equivalence relation:

$$< E > \geq < F > \Leftrightarrow \mathrm{Null}_E \subseteq \mathrm{Null}_F$$
$$< E > \otimes < F > = < E \otimes F >$$
$$< E > \oplus < F > = < E \oplus F >$$

Note that if $< E > \geq < F >$ then X is E-local $\Rightarrow$ X is F-local and so there is a natural transformation $L_E \to L_F$ which is obtained from id $\to L_F$ by applying L_E.

Theorem 1.13. *(Ohkawa) The collection of Bousfield classes forms a set of cardinality at least* $2^{\aleph_0}$ *and at most* $2^{2^{\aleph_0}}$.

We won't be needing this cool theorem anywhere in our talk.

2. Smashing Localization

Definition 2.1. A localization functor L is called smashing if it commutes with set-indexed direct sums or, equivalently, if the natural transformation $LX \to L\mathbb{S} \otimes X$ is an equivalence for all spectra X (i.e., $LX \simeq L\mathbb{S} \otimes X$). Further, L is finite, if there exists a collection of finite spectra that generates the category $\ker(L)$ of L-acyclics.

A Bousfield functor $L : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}$ is called smashing if it preserves colimits. If every L-acyclic spectrum is a colimit of compact L-acyclic spectra, then L is called finite and is in particular smashing.

Definition 2.2. If G is an abelian group, then the *Moore spectrum* SG is particular spectrum with properties such that,

(i) $\pi_0(SG) = G$,

(ii) $\pi_{<0}(SG) = 0$,
(iii) $H_{>0}(SG;\mathbb{Z}) = 0$.

A special case when $G = \mathbb{Z}/p\mathbb{Z}$ is called *mod-p Moore spectrum*.

Let SG be any Moore spectrum. We will localize a spectrum X with respect to SG so we can yield a spectrum $L_{SG}X$. An important example is localization at the Moore spectrum $\mathbb{S}/p$, also called as *p-completion*, in which we will be interested form now. For any spectrum X, X is E-local if and only if $[A, X] \simeq 0$ as in Lemma 1.5. A spectrum A is acyclic if $A \otimes \mathbb{S}/p = 0$. As taking colimits commutes with smash products, this is equivalent to saying that the cofiber of the map $A \otimes \mathbb{S} \xrightarrow{\mathrm{id}\otimes(\times p)} A \otimes \mathbb{S}$ is trivial. In other words, A being acyclic means the multiplication by p map $A \xrightarrow{\times p} A$ is an isomorphism in $\mathrm{ho}(\mathcal{S}\mathrm{p})$.

2.1. **p-completeness.** Observing Definition 2.1, the key points are that there is a map $\mathbb{S} \to L\mathbb{S}$ for any X then $X \to L\mathbb{S}$ is a localization and the local objects are closed under homotopy colimits. The later statement in a consequence of the previous statement, because

$$L\mathbb{S} \otimes \operatorname{colim} X_i \to \operatorname{colim}(L\mathbb{S} \otimes X_i)$$

is always an equivalence and the former is always local. Conversely, the homotopy-colimit preserving functors on spectra are all equivalent to functors of the form $X \to A \otimes X$ for some A, and the localization map $\mathbb{S} \to A$ is as we need it to be.

2.3. We will elaborate about *p-completeness* roughly.

Let us consider S to be the collection of multiplication by p maps $S^n \to S^n$ for $n \in \mathbb{Z}$, S-local spectra are those whose homotopy groups are $\mathbb{Z}[1/p]$-modules, and their equivalences are those maps which induce isomorphisms on homotopy groups after inverting p. The localization of $\mathbb{S}$ is the homotopy colimit

$$\mathbb{S}[1/p] = \operatorname{colim}\left(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \cdots\right),$$

which is a Moore spectrum for $\mathbb{Z}[1/p]$. We also see that $X \to \mathbb{S}[1/p] \otimes X$ is an S-equivalence for all X. Now we restrict S by considering it as a set of multiplication by m maps, where m is relatively prime to p, this replaces the ring $\mathbb{Z}[1/p]$ with the local ring $\mathbb{Z}_{(p)}$.

Let us consider a spectrum X is local for the maps $\mathbb{S}[1/p] \otimes S^n \to *$ if and only if the homotopy limit

$$\lim(\cdots \xrightarrow{p} X \xrightarrow{p} X) \cong \mathrm{Map}(\mathbb{S}[1/p], X)$$

is weakly contractible. Looking at the fiber sequence diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X & \xrightarrow{p} & X & \xrightarrow{p} & X \\
 & & \downarrow{\scriptstyle p^2} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} \\
\cdots & \longrightarrow & X & \xrightarrow{1} & X & \xrightarrow{1} & X \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & X/p^2 & \longrightarrow & X/p & \longrightarrow & *
\end{array}$$

shows that X is local if and only if the map $X \to X_p^\wedge = \lim_n X/p^n$ is an equivalence. We refer to a spectrum local for these maps as p-complete; a Bousfield localization of X will be called the p-completion; a trivial object is called p-adically trivial; an equivalence is called p-adic equivalence.

Now taking colimits from the above diagram, we get a fiber sequence

$$\Sigma^{-1}\mathbb{S}/p^{\infty} \to \mathbb{S} \to \mathbb{S}[1/p],$$

We can write

$$\mathbb{S}/p^{\infty} \cong \operatorname*{colim}_{n} \mathbb{S}/p^{n}$$

so that

$$X_p^{\wedge} \cong \mathrm{Map}(\Sigma^{-1}\mathbb{S}/p^{\infty}, X).$$

Moreover, the map $X_p^{\wedge} \to (X_p^{\wedge})_p^{\wedge}$ is always an equivalence. Therefore, $X_p^{\wedge}$ is always p-complete.

If multiplication by p is an equivalence on Y, then $Y \cong Y \otimes \mathbb{S}[1/p]$, and so maps $Y \to X$ are equivalent to maps $Y \to \mathrm{Map}(\mathbb{S}[1/p], X)$. For any X which is p-adically complete, this is trivial, so such objects Y are p-adically trivial. In particular, the fiber of $X \to X_p^{\wedge}$ is always trivial and so $X \to X_p^{\wedge}$ is a p-adic equivalence. Therefore this is a p-adic completion.

If each homotopy group of X has a bound on the order of p-power torsion, we can further identify the homotopy groups of $X_p^{\wedge}$ as the ordinary p-adic completions of the homotopy groups of X; if the homotopy groups of X are finitely generated, then $\pi_*(X_p^{\wedge}) \to \pi_*(X) \otimes \mathbb{Z}_p$.

2.2. **Rationalization.** Let us consider S as defined before, but now we enlarge it by considering the collection of multiplication by m maps $S^n \to S^n$ for $m > 0$. A spectrum X is S-local if and only if multiplication by m is an isomorphism on the homotopy groups $\pi_* X$ for all m, or equivalently if the maps $\pi_* X \to \pi_* X \otimes \mathbb{Q}$ are isomorphism, Such spectra are called *rational*.

Slogan 2.4. Due to Hurewicz and Serre, it is said to be that Rational homotopy theory is easy and as a consequence Rational stable homotopy theory is very easy.

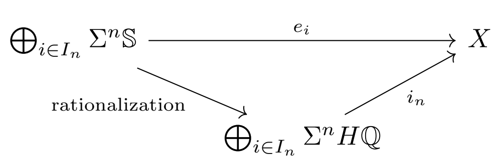
Recall 2.5. $\pi_*\mathbb{S}_{\mathbb{Q}} = \pi_*\mathbb{S} \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if} * = 0 \\ 0 & \text{otherwise.} \end{cases}$

So $\mathbb{S}_{\mathbb{Q}} = H\mathbb{Q}$.

Proposition 2.6. *Let X be a rational spectrum, then there exists an equivalence*

$$X \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^n H\pi_n X.$$

Proof. Since X is rational, we know that $\pi_n X$ is rational vector space for every n. We now have a basis $\{e_i\}_{i \in I_n}$, then we have a map



And $\Sigma^n H\pi_n X = \bigoplus_{i \in I_n} \Sigma^n H\mathbb{Q}$. The map i_n is an isomorphism to π_n and zero otherwise. And $\bigoplus_n i_n$ is an isomorphism on homotopy and therefore an equivalence. $\square$

2.2.1. *Chern character.* The Chern character is a ring homomorphism

$$\mathrm{ch} : K^*(-) \to H^*(-;\mathbb{Q}) \otimes K^*(*)$$

which induces an isomorphism $K^*(-) \otimes \mathbb{Q} \simeq H^*(-;\mathbb{Q}) \otimes K^*(*)$. In other words,

Corollary 2.7. *(Chern character) Let X be a finite space, there is an equivalence*

$$KU^0(X)_{\mathbb{Q}} \simeq \bigoplus_{n\in\mathbb{Z}} H^{2n}(X;\mathbb{Q}).$$

Proof. Since X is a finite space, $\mathrm{Map}(\Sigma^\infty X_+, -)$ commutes with filtered colimits. So we have

$$\begin{aligned} KU^0(X) \otimes \mathbb{Q} &= \pi_0\mathrm{Map}(\Sigma^\infty X_+, KU) \otimes \mathbb{Q} \\ &= \pi_0\mathrm{Map}(\Sigma^\infty X_+, KU \otimes \mathbb{Q}) \\ &= \pi_0\mathrm{Map}\big(\Sigma^\infty X_+, \bigoplus_{n\in\mathbb{Z}} \Sigma^{2n} H\mathbb{Q}\big) \\ &= \bigoplus_{n\in\mathbb{Z}} \pi_0\mathrm{Map}(\Sigma^\infty X_+, \Sigma^{2n} H\mathbb{Q}) \\ KU^0(X)_{\mathbb{Q}} &= \bigoplus_{n\in\mathbb{Z}} H^{2n}(X;\mathbb{Q}). \end{aligned}$$

□

The Chern character on complex K-theory is a map from complex oriented cohomology theory of chromatic height 1 to ordinary cohomology of chromatic level 0. For more on higher Chern characters refer[HSS17]

Sullivan in his [Sul05] studied about arithmetic fracture techniques that allowed simply-connected space X to be recovered from its rationalization $X_{\mathbb{Q}}$ and its p-adic completions $X_p^\wedge$ via a homotopy pullback diagram:

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \big\downarrow & & \big\downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \big(\prod_p X_p^\wedge\big)_{\mathbb{Q}} \end{array}$$

This allows us to reinterpret homotopy theory. We are not using p-adic completion and rationalization to understand algebraic invariant of X; but rather, knowledge of X is equivalent to knowledge of its localization, completions and an "arithmetic attaching map"$(X_{\mathbb{Q}} \to \big(\prod_p X_p^\wedge\big)_{\mathbb{Q}})$.

Theorem 2.8. *Let X be a simply connected space whose homotopy groups are finitely generated. Then the above diagram is a homotopy pullback square.*

Proof. Let F denote the fiber product

$$\big(\prod_p X_p^\wedge\big) \times_{\left(\prod_p X_p^\wedge\right)_{\mathbb{Q}}} X_{\mathbb{Q}},$$

so we have a canonical map $\alpha : X \to F$ and which to show this is a homotopy equivalence. The homotopy groups of F lies in a last exact sequence

$$\cdots \pi_n F \to \pi_n X_{\mathbb{Q}} \times \pi_n\big(\prod_p X_p^\wedge\big) \xrightarrow{\beta} \pi_n\big(\prod_p X_p^\wedge\big)_{\mathbb{Q}} \to \cdots$$

Let A denote $\pi_n X$, then the map β can be identified as $\beta : A_{\mathbb{Q}} \times \prod_p A_p \to (\prod_p A_p)_{\mathbb{Q}}$. From the properties of p-completion this map is a surjective. The long exact sequence can be broke into short exact sequences, and gives isomorphisms

$$\pi_n F \simeq \ker(\beta) \simeq A.$$

These isomorphisms are induced by the map $A \to \pi_n X \to \pi_n F$, so that α is a homotopy equivalence.

□

We can also consider a completion at all primes at once. Then we say that a spectrum is *profinitely complete* if it is local for $\bigoplus_p \mathbb{S}/p$ where p ranges through all primes. So, a spectrum X is profinitely acyclic if and only if $X/p = 0$ for every p.

2.3. **KU-localization.** KU represents the topological K-theory. We will use arithmetic fracture squares to simply it to a study of $(L_{KU}X)_{\mathbb{Q}}$ and $(L_{KU}X)^\wedge_p$ for every p.

Lemma 2.9. *Let X be a spectrum and E be a spectrum such that $E_{\mathbb{Q}} \neq 0$. Then the map*

$$X \to L_E X$$

is a rational equivalence. In particular we have $(L_E X)_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$ and every rational spectrum is KU-local.

Proof. We need show that any E-acyclic spectrum is rationally trivial. Let $A = \mathrm{fib}(X \to L_E X)$, this is a E-acyclic spectrum such that $E \otimes A = 0$. So,

$$E \otimes A = 0 = (E \otimes A)_{\mathbb{Q}} = E_{\mathbb{Q}} \otimes A$$

we know that $E_{\mathbb{Q}} = \bigoplus_{n\in\mathbb{Z}} \Sigma^n H\mathbb{Q} \neq 0$. So the right hand side is a direct sum of shifted copies of $H\mathbb{Q} \otimes A \simeq A_{\mathbb{Q}}$. By assumption $E_{\mathbb{Q}} \neq 0$, so $A_{\mathbb{Q}} = 0$. □

Lemma 2.10. *Let E and X be a spectrum. Then $X^\wedge_p$ is E-local if and only if X/p is E-local.*

Proof. We know that $X^\wedge_p/p \simeq X/p$, if $X^\wedge_p$ is E-local, then so it X/p. Conversely, if X/p is E-local, we have a fiber sequence

$$X/p \to X/p^n \to X/p^{n-1},$$

If X/p is E-local, X/p^n is E-local $\forall n$. Since, $X^\wedge_p \simeq \lim_n X/p^n$ is E-local. □

Lemma 2.11. *Let E and X be spectrum. Then $(L_E X)^\wedge_p \simeq L_{E/p} X$.*

Proof. The map $X \to (L_E X)^\wedge_p$ is an E/p-equivalence, because

$$X \xrightarrow{E-\mathrm{equiv}} L_E X \xrightarrow{\mathbb{S}/p-\mathrm{equiv}} (L_E X)^\wedge_p$$

From the previous lemma, we can say that $(L_E X)^\wedge_p$ is a E-local.

Now let us consider A a spectrum that is E/p-acyclic (i.e, $E \otimes A/p = 0$). So, $\mathrm{Map}(A/p, (L_E X)^\wedge_p) = 0$. This gives us a map

$$p : \mathrm{Map}(A/p, (L_E X)^\wedge_p) \xrightarrow{\simeq} \mathrm{Map}(A, (L_E X)^\wedge_p)$$

is an equivalence. Since, $\mathrm{Map}(A, (L_E X)^\wedge_p)$ is p-complete

$$\lim_n \mathrm{Map}(A, L_E X/p^n) = \lim_n \mathrm{Map}(A, L_E X)/p^n$$

$\Rightarrow \mathrm{Map}(A, (L_E X)^\wedge_p) = 0.$ □

Recollecting all along with fracture square, we have a pullback square

$$\begin{array}{ccc} L_{KU}X & \longrightarrow & \prod_p L_{KU/p}X \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p L_{KU/p}X)_{\mathbb{Q}} \end{array}$$

The following results are from [Bou79]. We will state some results Adams-Baird, but won't prove anything.

2.12. (i) (Adams operations) There exists a map of ring spectra $\psi^r : KU_p^\wedge \to KU_p^\wedge$ for all r coprime with p. This gives an action of $\mathbb{Z}_p^\times$ on $KU_p^\wedge$.

(ii) (Self-maps)Let p be odd prime(just for this point), there exists a self map v_1 on $\mathbb{S}/p$ as $v_1 : \Sigma^{2(p-1)}\mathbb{S}/p \to \mathbb{S}/p$

Let us denote the Bousfield localization with respect to $\mathbb{S}[1/p]$ as $T(0)$. From the point of view of Chromatic homotopy theory, localization at $T(0)$ is just the zeroth in a whole sequence of localizations depending on a fixed prime p. By periodicity theorem of Hopkins-Smith [HS98], there exists higher analogous of multiplication by p: A self-map v_1 of $\mathbb{S}/p$, a self-map v_2 of $\mathbb{S}/(p, v_1)$, and in general a self-map v_{n+1} of $\mathbb{S}/(p, v_1, \ldots, v_n)$. We will denote

$$T(n) = \mathbb{S}/(p, v_1, \ldots, v_{n-1})[v_n^{-1}]$$

as the telescope of a v_n self-map and $L_{T(n)}$ denotes its Bousfield localization functor, and refer to $L_{T(n)}$ as *telescopic localization* (at height n). A result of Waldhausen [Wal84] shows that v_1 is a KU-equivalence. v_1 is constructed in such as way that,

$$\begin{array}{ccc} \Sigma^{2(p-1)}\mathbb{S}/p & \xrightarrow{v_1} & \mathbb{S}/p \\ \uparrow & & \downarrow \\ \Sigma^{2(p-1)}\mathbb{S} & \longrightarrow & \Sigma\mathbb{S} \end{array}$$

Moreover ψ^r is a map of rings, then there exists a nullhomotopy of the composition

$$\mathbb{S}_p^\wedge \to KU_p^\wedge \xrightarrow{\psi^r - 1} KU_p^\wedge$$

Therefore $\mathbb{S}/p \to KU/p \xrightarrow{\psi^r - 1} KU/p$ is a nullhomotopy as well. We get

$$\begin{array}{ccccc} \Sigma^{2(p-1)}\mathbb{S}/p & \longrightarrow & \Sigma^{2(p-1)}KU/p & \longrightarrow & \Sigma^{2(p-1)}KU/p \\ \simeq\downarrow v_1 & & \simeq\downarrow v_1 & & \simeq\downarrow v_1 \\ \mathbb{S} & \longrightarrow & KU/p & \longrightarrow & KU/p \end{array}$$

This is an equivalence since v_1 is a KU-equivalence. Now we define

$$\mathbb{S}/p[v_1^{-1}] = \operatorname{colim}(\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-2(p-1)}\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-4(p-1)}\mathbb{S}/p \xrightarrow{v_1} \Sigma^{-6(p-1)}\mathbb{S}/p \xrightarrow{v_1} \cdots)$$

So we get a sequence Such that $\mathbb{S}/p[v_1^{-1}]$ is the fiber of the map $\psi^r - 1 : KU/p \to KU/p$.

Similarly, when p is even prime (2), an equivalent statement hold, where we replace v_1 with a map $v_1^4 : \Sigma^8\mathbb{S}/2 \to \mathbb{S}/2$ such that $\mathbb{S}/2[v_1^{-4}]$ is the fiber of a map $KO/2 \to KO/2$.

This is due to a computational statement (described below) and properties of *image of J*.

Theorem 2.13. *(Mahowald and Miller) The groups $\pi_i\mathbb{S}/2[v_1^{-4}]$ have order:*

$$\begin{cases} 4 & \text{if } i \equiv 0 \mod 8 \ \& \ \equiv 3 \mod 8 \\ 8 & \text{if } i \equiv 1 \mod 8 \ \& \ \equiv 2 \mod 8 \\ 2 & \text{if } i \equiv 4 \mod 8 \ \& \ \equiv 7 \mod 8 \\ 1 & \text{otherwise} \end{cases}$$

For a odd prime p, the groups $\pi_i\mathbb{S}/p[v^{-1}1]$ have orders:

$$\begin{cases} p & \text{if } i \equiv 0 \mod 2(p-1) \ \& \ \equiv -1 \mod 2(p-1) \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.14. *Let X be a spectrum. Then X/p is KU-local if and only if the map $v_1 : \Sigma^{2(p-1)}X/p \to X/p$ is an equivalence. Further, we have $L_{KU}X/p \simeq X/p[v_1^{-1}]$.*

Proof. Since v_1 is KU-equivalence, if X/p is KU-local then multiplication by v_1 is an equivalence as well. Conversely, if v_1 is an equivalence then we have an equivalence

$$X/p \simeq X/p[v_1^{-1}] = X \otimes \mathbb{S}/p[v_1^{-1}]$$

But by earlier arguments, we can identify $X \otimes \mathbb{S}/p[v_1^{-1}]$ as a fiber of a map $X \otimes KU/p \to X \otimes KU/p$. So it it enough to prove that $X \otimes KU/p$ is a KU-local. It is a KU-module, so definitely KU-local. Finally the map $X/p \to X/p[v_1^{-1}]$ is a KU-equivalence since it is a composition of KU-equivalence and therefore it is a KU-localization map since the target is KU-local. □

Definition 2.15. The map $\pi_i(O(n)) \to \pi_{n+i}(S^n)$ is called the *J-homomorphism*. We are mostly interested in the stable version, so can be rewritten as $J : \pi_*(O) \to \pi_*\mathbb{S}$.

We will denote J_p for the p-complete image of J spectrum, and the spectrum $L_{KU}\mathbb{S}$ is also sometimes called the image of J spectrum. This is because the map $\pi_*\mathbb{S} \to \pi_*L_{KU}\mathbb{S}$ is split surjection for $* > 0$ and it identifies $\pi_*L_{KU}\mathbb{S}$ with the image of the J-homomorphism.

Theorem 2.16. *The KU-localization is smashing, that is the map*

$$L_{KU}X \to L_{KU}\mathbb{S} \otimes X$$

is an equivalence. Moreover we can write a pullback square

$$\begin{array}{ccc} L_{KU}\mathbb{S} & \longrightarrow & \prod_p J_p \\ \downarrow & & \downarrow \\ H\mathbb{Q} & \longrightarrow & \left(\prod_p J_p\right)_{\mathbb{Q}} \end{array}$$

where J_p is the fiber of the map $\psi^r - 1 : KU_p^\wedge \to KU_p^\wedge$ for p odd, and the same for $p = 2$ by replacing KU with KO.

Proof. We know that the map $X \to L_{KU}\mathbb{S} \otimes X$ is always a KU-equivalence, so it is enough to prove that the target is KU-local. From the arithmetic square and every rational spectrum is KU-local, it is again reduced to show that $(L_{KU}\mathbb{S} \otimes X)_p^\wedge$ is KU-local for every prime, that is that $L_{KU}\mathbb{S}/p \otimes X$ is KU-local. By the previous corollary, we have

$$L_{KU}\mathbb{S}/p \otimes X \simeq \mathbb{S}/p[v_1^{-1}] \otimes X \simeq X/p[v_1^{-1}] \simeq L_{KU}(X/p)$$

and therefore the left hand side is KU-local.

Now we are left to prove that $(L_{KU}\mathbb{S})_p^\wedge \simeq J_p$. Let $\eta : \mathbb{S} \to KU$ be the unit. Since $\psi^r : KU_p^\wedge \to KU_p^\wedge$ is a map of rings, we have $\psi^r\eta \simeq \eta$, therefore we can choose a nullhomotopy of $(\psi^r - 1) \circ \eta$. So we have a map $\mathbb{S} \to J_p$, moreover J_p is KU-local and p-complete, so we get

$$(L_{KU}\mathbb{S})_p^\wedge \to J_p.$$

This is an equivalence after tensoring by $\mathbb{S}/p$, and so it is an equivalence. □

3. Morava K-theory

The most interesting periodic self-maps occur when X is a finite p-torsion complex. Also Morava found that working with $MU_*(-)$ is tedious and provided an algebraic setting $K(n)_*(-)$ which is easy to work with. These are due to unpublished works of Morava and discussed in [JW75].

Proposition 3.1. *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geq 0$ with the following properties.*

(i) $K(0)_*(X) = H_*(X;\mathbb{Q})$ *and* $\overline{K(0)_*}(X) = 0$ *when* $\overline{H_*}(X)$ *is all torsion.*
(ii) $K(1)_*(X)$ *is one of $p-1$ isomorphic summands of* $\bmod p$ *complex K-theory.*
(iii) $K(0)_*(pt) = \mathbb{Q}$ *and for $n > 0$,* $K(n)_*(pt) = \mathbb{Z}/(p)[v_n, v_n^{-1}]$ *where the dimension of v_n is $2(p^n - 1)$. This ring is graded field inn the sense that every graded module over it is free. $K(n)_*(X)$ is a module over $K(n)_*(pt)$.*
(iv) *There is a Künneth isomorphism*

$$K(n)_*(X \times Y) \simeq K(n)_*(X) \otimes_{K(n)_*(pt)} K(n)_*(Y)$$

. This is what makes is easy to work with $K(n)_*(X)$ than $MU_*(X)$.
(v) *Let X be a p-local finite spectrum. If $\overline{K(n)_*}(X)$ vanishes, then so does $\overline{K(n-1)_*}(X)$*
(vi) *For X a p-local finite spectrum and n sufficiently large enough, we have*

$$\overline{K(n)_*}(X) = K(n)_*(pt) \otimes \overline{H_*}(X, \mathbb{Z}/(p)).$$

Due to Quillen's theorem on formal group law, we have $\pi_*MU_{(p)} \simeq L_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, t_2, \dots]$, where we may assume that $v_i = t^{p^i - 1}$ for each $i > 0$. By convention, we set $t_0 = p \in \pi_0 MU_{(p)}$.

For each integer k, we denote $M(k)$ to be the cofiber of the map $\Sigma^{2k}MU_{(p)} \to MU_{(p)}$ given by multiplication by t_k.

Definition 3.2. [Lur10] For a prime p and $n > 0$. We denote $K(n)$ denote the smash product (over $MU_{(p)}$) of $MU_{(p)}[v_n^{-1}]$ with $\bigotimes_{k \neq p^n - 1} M(k)$. The spectrum $K(n)$ is called Morava K-theory.

Since each $M(k)$ admits a unital and homotopy associative multiplication, we see that $K(n)$ also has the structure of a homotopy associative $MU_{(p)}$-algebra; if $p \neq 2$, we can even assume that K is homotopy commutative.

The homotopy groups of $K(n)$ are given by

$$\pi_*K(n) \simeq (\pi_*MU_{(p)})[v_n^{-1}]/(t_0, t_1, \dots, t_{p^n-2}, t_{p^n}, \dots) \simeq \mathbb{F}_p[v_n^{\pm}],$$

where v_n has degree $2(p^n - 1)$.

We have a map of ring spectrum $MU_{(p)} \to K(n)$, giving a complex orientation on $K(n)$. This determines a formal group law over the ring $\pi_* K(n) \simeq \mathbb{F}_p[v_n^{\pm}]$, which has height exactly n.

In other words Definition 3.2 can be interpreted as, a spectrum $K(n)$ is n-th Morava K-theory with coefficients $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$, where v_i is of degree $2(p^i - 1)$.

There exist something called n-th Johnson-Wilson theory $E(n)$ with coefficients $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n][v_n^{-1}]$, where degree of $v_i = 2(p^i - 1)$.

3.3. From Proposition 3.1 we can say that, if F is a finite spectrum that is $K(n)$-acyclic, then it is also $K(n-1)$-acyclic; since $< E(n) >=< \bigvee_{i=0}^{n} K(i) >$, so the spectrum F is also $E(n)$-acyclic.

Definition 3.4. We say a p-local finite spectrum X has *type* n if $K(n)_*(X) \neq 0$ but $K(m)_*(X) \simeq 0$ for $m < n$. For example, X has type 0 if $H_*(X; \mathbb{Q}) \simeq 0$, or equivalently if $H_*(X; \mathbb{Z})$ is not a torsion group.

Every nonzero finite p-local spectrum X has *type* n for some unique n.

As we saw in 2.2, by the periodicity theorem in [HS98], any finite type n spectrum F admits an v_n self-map, and we write $T(n) = F[v_n^{-1}]$ for the associated telescope. It is due to the thick subcategory theorem that the Bousfield class of $T(n)$ depends only on n.

Definition 3.5. Let $n \geq 0$, then we define two localization functors on the stable homotopy category by

(i) $L_n^f = L_{T(0) \vee T(1) \vee \cdots \vee T(n)}$ representing the finite L_n-localization and,
(ii) $L_n = L_{E(n)} \simeq L_{K(0) \vee K(1) \cdots \vee K(n)}$ representing the L_n-localization for a particular n.

The functors L_n^f are in fact finite localizations, with $\ker(L_n^f)$ generated by any finite type $(n+1)$ spectrum.

3.6. Before we get any further, Let's take a look at a particular case. We stated that for every integer $n \geq 0$, there exists a finite p-local spectrum X of type n. If $n = 0$, this just means that the rational homology $H_*(X; \mathbb{Q})$ is non-zero. We can obtain this by taking X to be the p-local $\mathbb{S}/p$.

When $n = 1$, we can define X to be a mod p Moore spectrum, which is defined by the cofiber sequence

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \to X.$$

The multiplication p annhilates $K(1)_*(\mathbb{S}) \simeq \mathbb{F}_p[v_1^{-1}]$, the map $K(1)_*(\mathbb{S}) \to K(1)_*(X)$ is injective. In particular, $K(1)_*(X) \neq 0$, so that X has type 1.

3.1. **Telescopic Conjecture.** We will discuss about some results for which we won't prove anything, but would recommend to refer [Rav92b].

It is from the thick subcategory theorem that any finite localization functor of the category of spectra which is not equal to the identity or the zero functor must be on of the L_n^f.

Proposition 3.7. *For each n, the functor L_n^f is a finite and thus smashing localization. If F is finite type n spectrum, then $L_n^f F \simeq T(n)$.*

Theorem 3.8. *For every $n \geq 0$, the localization functor $L_{E(n)}$ is smashing.*

There is a natural transformation $L_n^f \to L_n$ which is an equivalence on all MU-module spectra and all L_i-local spectra for any $i \geq 0$. In other words, there is a close relationship between the functors L_n and their counterpart L_n^f. If two localization were in fact equivalent for all n, then two naturally arising filtrations on the stable homotopy groups of spheres would coincide, making the computation of $\pi_* S^0$ more computable using algebraic techniques. This idea gave raise to

Conjecture 3.9. *For any $n \geq 0$, the natural map $L_n^f \to L_n$ is an equivalence.*

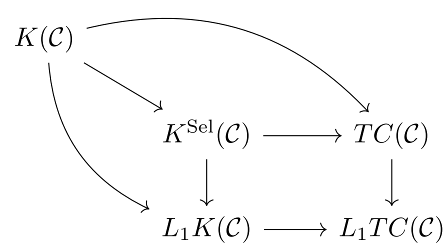
For $n = 0$, both L_0^f and L_0 identify with rationalization. And using Theorem 2.13 of Mahowald and Miller, Bousfield was able to state; that the telescopic conjecture holds at height $n = 1$.

Theorem 3.10. *For $n = 1$, the natural map $L_{K(1)} \to (L_{KU})_p^\wedge$ is an equivalence (i.e., $L_{K(1)} = (L_{KU})_p^\wedge$).*

Proof. The proof directly follows from 3.6 and Corollary 2.14. Therefore saying $L_{K(1)}$ is a telescopic localization. □

4. Selmer K-theory

Definition 4.1. Let $\mathcal{C}$ be a small stable ∞-category. The *Selmer K-theory* $K^{\mathrm{Sel}}(\mathcal{C})$ is defined to the homotopy pullback



$$K^{\mathrm{Sel}}(\mathcal{C}) = L_1K(\mathcal{C}) \times_{L_1TC(\mathcal{C})} TC(\mathcal{C}). \tag{4.2}$$

Applying the localization natural transformation $\mathrm{id} \to L_1$ to the cyclotomic trace $K \to TC$ gives rise to a natural map $K \to K^{\mathrm{Sel}}$ which factors both the trace $K \to TC$ and the localization $K \to L_1K$.

Example 4.3. [CM21](Rational Selmer K-theory) In Lemma 2.9 we show that $X_{\mathbb{Q}} \to (L_1X)_{\mathbb{Q}}$ is an equivalence for any spectrum X and any rational spectrum is KU-local.

Hence when we apply rationalization to K^{Sel}(to the above formula 4.2) is simply the rationalization of K; that is, $K^{\mathrm{Sel}} \otimes \mathbb{Q} = K \otimes \mathbb{Q}$ and rational algebraic K-theory is as mysterious as ever.

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