SoSe 24 Algebraic Topology II
Exercise sheet 1 (due May 3)

Exercise 1.1. Let X be a pointed space.

(a) Show that $\Sigma X = S^1 \wedge X$ is a cogroup object in hTop_*.

(b) Show that $\Omega X = \mathrm{Map}_*(S^1, X)$ is a group object in hTop_*.

Exercise 1.2.

(a) Show that $S^{n-1} \hookrightarrow D^n$ is a cofibration.

(b) Deduce that any relative CW complex $A \hookrightarrow X$ is a cofibration.

Exercise 1.3. Let $i\colon A \hookrightarrow X$ and $j\colon B \hookrightarrow Y$ be cofibrations. Show that $i \times j\colon A \times B \hookrightarrow X \times Y$ is a cofibration

Exercise 1.4.

(a) If A is contractible and $A \hookrightarrow X$ is a cofibration, show that the quotient map $X \to X/A$ is a homotopy equivalence.

(b) Deduce that if $i\colon A \hookrightarrow X$ is a cofibration, then $\mathrm{Cone}(i) \to X/A$ is a homotopy equivalence.

Exercise 1.5. Recall that a pointed space (X, x_0) is *well-pointed* if the inclusion of the base point $\{x_0\} \hookrightarrow X$ is a cofibration. Let $f\colon X \to Y$ be a pointed map with X and Y well-pointed. Show that if f is an unpointed homotopy equivalence, then it is a pointed homotopy equivalence.

Exercise 1.6. Let A be a subspace of X and let $x_0 \in A$. For $n \geq 1$, define the relative homotopy group $\pi_n(X, A, x_0)$ as follows:

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, *), (X, A, x_0)].$$

It is only a pointed set for $n = 1$. For $n \geq 2$, the pinch map defines a group structure.

(a) Let F be the homotopy fiber of the inclusion $i\colon A \hookrightarrow X$ at x_0. Show that there is a bijection $\pi_n(X, A, x_0) \cong \pi_{n-1}(F, x_0)$ and convince yourself that it is an isomorphism of groups when $n \geq 2$.

The long exact sequence for $A \hookrightarrow X$ can therefore be written as

$$\cdots \to \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{\phi} \pi_n(X, A, x_0) \to \cdots.$$

(b) Identify the maps ∂ and ϕ in terms of the definition of $\pi_n(X, A, x_0)$.

Exercise 1.7. The *Hopf fibration* $\eta\colon S^3 \to S^2$ is the restriction to S^3 of the canonical projection $\mathbb{C}^2 - 0 \to \mathbb{CP}^1 = S^2$.

(a) Show that η is a fiber bundle with fiber S^1.

(b) Deduce that $\pi_2(S^2) \cong \mathbb{Z}$.
(You should assume the connectivity of spheres: $\pi_k(S^n) = 0$ for $k < n$.)

Later, we will be able to deduce from this that $\pi_n(S^n) \cong \mathbb{Z}$ for all higher n. Moreover, one can show that $\pi_n(S^n)$ is generated by the identity of S^n.

(c) Assuming this, show that $\pi_3(S^2) = \mathbb{Z}\eta$.

(d) Devise a quaternionic version of the Hopf map, called ν. Argue that ν has infinite order in the homotopy group $\pi_7(S^4)$.

Remark. There is further an octonionic version of the Hopf map, called σ, that has infinite order in $\pi_{15}(S^8)$, but its construction is more subtle.