SoSe 24 Algebraic Topology II
Exercise sheet 2 (due May 17)

**Exercise 2.1.** Let $f\colon X \to Y$ be a pointed map with homotopy fiber $F$, and consider the exact sequence of pointed sets

$$\pi_1(Y) \xrightarrow{\partial} \pi_0(F) \to \pi_0(X).$$

Construct an action of $\pi_1(Y)$ on the set $\pi_0(F)$ such that:

- $\partial$ is a $\pi_1(Y)$-equivariant map;
- two elements $a, b \in \pi_0(F)$ have the same image in $\pi_0(X)$ iff there exists $\gamma \in \pi_1(Y)$ such that $\gamma a = b$.

**Exercise 2.2.** Let $X$ and $Y$ be pointed CW complexes. Suppose that $X$ is $m$-connected and $Y$ is $n$-connected. How connected are the following spaces?

(a) $X \vee Y$

(b) $X \times Y$

(c) $X \wedge Y$

*Hint.* Recall that a CW complex is $n$-connected iff it is homotopy equivalent to one with no cells in dimensions $\leq n$.

**Exercise 2.3.** Let $A$ be a CW complex of dimension $\leq n$ and let $f\colon X \to Y$ be an $(n-1)$-connected map. Show that the induced map

$$f_*\colon [A, X] \to [A, Y]$$

is surjective, and that it is even bijective if $f$ is $n$-connected.

*Hint.* Use the compression lemma.

**Exercise 2.4.** Let $X$ and $Y$ be pointed spaces.

(a) Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ admits a *section* up to pointed homotopy after taking loops.

(b) Deduce that, for $n \geq 2$,

$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

(c) Assume for simplicity that $X$ and $Y$ are CW complexes. What are the minimal connectivity assumptions on $X$ and $Y$ guaranteeing that the canonical map $\pi_{n+1}(X \times Y, X \vee Y) \to \pi_{n+1}(X \wedge Y)$ is an isomorphism?

**Exercise 2.5.** Let $n \geq -2$ and let $X$ be any space. An *$n$-truncation* of $X$ is an $n$-connected map $X \to X_n$ where $X_n$ is $n$-truncated.

(a) Show that there exists a tower

$$\begin{array}{ccccccccc} & X & \searrow & \searrow & & & & & \\ & \downarrow & & & & & & & \\ \cdots \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow \cdots \longrightarrow & X_{-2}, \end{array}$$

where each map $X \to X_n$ is an $n$-truncation of $X$.

*Hint.* First construct an $n$-truncation $X \to X_n$ by attaching cells of dimensions $\geq n+2$. Then, show that the $n$-truncation factors through the $(n+1)$-truncation.

(b) Show that $n$-truncations are unique up to weak equivalence: If $t_n \colon X \to X_n$ and $t'_n \colon X \to X'_n$ are two $n$-truncations of $X$, then there exists a zigzag of weak equivalences under $X$ between $X_n$ and $X'_n$.

This is called the *Postnikov tower* of $X$ and one often writes $\tau_{\leq n} X$ for the $n$-truncation of $X$. Note that the homotopy fibers of $X_n \to X_{n-1}$ have no nontrivial homotopy groups except in degree $n$; such spaces are called *Eilenberg–Mac Lane spaces* of degree $n$, see Problem 1.5 below.

(c) Show that $S^2 = \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is a 2-truncation of $S^2$.

**Exercise 2.6.** (Eilenberg–Mac Lane spaces) Let $n \geq 1$ and let $A$ be a group (abelian if $n \geq 2$).

(a) Construct a pointed CW complex $K(A, n)$ such that

$$\pi_i(K(A,n)) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{otherwise,} \end{cases}$$

and show that any two CW complexes with this property are homotopy equivalent.

(b) Show that $[K(A,n), K(B,n)]_* \cong \operatorname{Hom}(A,B)$. In other words, the construction $A \mapsto K(A,n)$ is an equivalence of categories between (abelian) groups and the homotopy category of pointed Eilenberg–Mac Lane spaces of degree $n$.

**Exercise 2.7.**

(a) Compute the homotopy groups of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ in terms of the homotopy groups of spheres.

(b) Let $n \geq 2$. Compute the relative homotopy group $\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ and observe that the canonical map $\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \to \pi_n(\mathbb{RP}^n / \mathbb{RP}^{n-1})$ is not an isomorphism.