SoSe 24 Algebraic Topology II
Exercise sheet 3 (due May 31)

Exercise 3.1. A space X is called *acyclic* if $\tilde{H}_*(X) = 0$. Prove the following statements:

(a) If X is acyclic, then it is 0-connected and $\pi_1(X)$ is a perfect group (i.e., generated by commutators).

(b) If X is acyclic and 1-connected, then X is weakly contractible.

(c) X is acyclic if and only if its suspension $\mathrm{Susp}(X)$ is weakly contractible.

Remark. An example of an acyclic space is the 2-dimensional CW complex obtained from $S^1 \vee S^1$ by attaching two 2-cells with attaching maps a^5b^{-3} and $b^3(ab)^{-2}$ in $\pi_1(S^1 \vee S^1) = \langle a, b\rangle$. Its fundamental group is the so-called *binary icosahedral group*, which is a perfect group of order 120.

Exercise 3.2. Prove the homotopical Poincaré conjecture: every simply connected compact 3-manifold is homotopy equivalent to a 3-sphere.

Hint. Poincaré duality states that for a compact d-manifold M, $H_n(M, \tilde{\mathbb{Z}}) \cong H^{d-n}(M, \mathbb{Z})$, where $\tilde{\mathbb{Z}}$ is the orientation local system. You may also use the standard fact, due to Milnor, that any manifold is homotopy equivalent to a CW complex.

Exercise 3.3. Recall that the stable homotopy groups of a (well-pointed) space X are defined by

$$\pi_n^s(X) = \operatorname*{colim}_{k\to\infty} \pi_{k+n}(\Sigma^k X) \cong \pi_{k+n}(\Sigma^k X) \quad \text{for any } k \geq n+2,$$

where $\Sigma X = S^1 \wedge X$ and the transition map $\pi_{k+n}(\Sigma^k X) \to \pi_{k+1+n}(\Sigma^{k+1}X)$ is $[f] \mapsto [\Sigma f]$.

For $\alpha \in \pi_m^s(S^0)$ represented by $f\colon S^{k+m} \to S^k$ and $\beta \in \pi_n^s(S^0)$ represented by $g\colon S^{l+n} \to S^l$, define

$$\alpha\beta \in \pi_{m+n}^s(S^0)$$

as the element represented by

$$S^{k+l+m+n} \cong S^k \wedge S^l \wedge S^m \wedge S^l \xrightarrow{\tau} S^k \wedge S^m \wedge S^l \wedge S^n \cong S^{k+m} \wedge S^{l+n} \xrightarrow{f\wedge g} S^k \wedge S^l \cong S^{k+l},$$

where $\tau\colon S^l \wedge S^m \to S^m \wedge S^l$ is the swap isomorphism $(x, y) \mapsto (y, x)$.

(a) Let $\sigma \in \Sigma_n$ be a permutation. Show that the induced map $(S^1)^{\wedge n} \to (S^1)^{\wedge n}$ has degree $\mathrm{sgn}(\sigma)$.

(b) Show that the product $\alpha\beta$ is well-defined, i.e., independent of the choice of $k, l \gg 0$.

Hint. Denote by $f \star g$ the above composition representing $\alpha\beta$. One has to show that $f \star g$, $\Sigma f \star g$ and $f \star \Sigma g$ define the same element in $\pi_{m+n}^s(S^0)$. First note that $\Sigma(f \star g) = \Sigma f \star g$. Second, show that $\Sigma(f \star g)$ and $f \star \Sigma g$ are homotopic when k is even, using (a). Finally, deduce that $\Sigma^2(f \star g)$ and $\Sigma(f \star \Sigma g)$ are homotopic when k is odd.

(c) Show that this product makes $\pi_*^s(S^0)$ into a graded ring, i.e., that it is associative and distributes over addition.

Hint. For distributivity, you need to show that the supension of the pinch map of S^k ($k \geq 1$) is the pinch map of S^{k+1}. This can be checked either geometrically or by the Eckmann–Hilton argument.

(d) Show that this ring is graded-commutative: if $\alpha \in \pi_m^s(S^0)$ and $\beta \in \pi_n^s(S^0)$, then

$$\alpha\beta = (-1)^{mn}\beta\alpha.$$

Hint. Choose a representative $f\colon S^{k+m} \to S^k$ of α with k even.

Exercise 3.4. Let $\eta\colon S^3 \to S^2$ be the Hopf fibration (see Exercise 1.7).

(a) Define maps $f\colon S^3 \to S^3$ and $g\colon S^2 \to S^2$ with $\deg(f) = 1$ and $\deg(g) = -1$ such that $\eta \circ f = g \circ \eta$.

(b) Deduce that $\Sigma\eta \in \pi_4(S^3)$ is 2-torsion.

Hint. Use that $\pi_4(S^3) \cong \pi_1^s(S^0)$ and the ring structure on $\pi_*^s(S^0)$ from Exercise 3.

Exercise 3.5. Let $p, q \geq 1$. Define $h\colon S^{p+q-1} \to S^p \vee S^q$ by the commutativity of the following diagram:

$$\begin{tikzcd}
\partial I^{p+q} \arrow[r, hook] \arrow[rrd, dashed] & I^{p+q} = I^p \times I^q \arrow[r] & I^p/\partial I^p \times I^q/\partial I^q \\
 & & I^p/\partial I^p \vee I^q/\partial I^q. \arrow[u, hook]
\end{tikzcd}$$

One can show that the maps h define a structure of *graded Lie coalgebra* on the graded cogroup $\{S^{n+1}\}_{n\geq 0}$ in the pointed homotopy category. Equivalently, the homotopy groups $\pi_{*+1}(X)$ of a pointed space X have the structure of a graded Lie algebra, with bracket $[-,-]\colon \pi_p(X) \times \pi_q(X) \to \pi_{p+q-1}(X)$ given by:

$$[\alpha,\beta]\colon S^{p+q-1} \xrightarrow{h} S^p \vee S^q \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\nabla} X.$$

This bracket is called the *Whitehead product.*

(a) What is the homotopy cofiber of h?

(b) Describe $[-,-]\colon \pi_1(X) \times \pi_n(X) \to \pi_n(X)$ in terms of the action of $\pi_1(X)$ on $\pi_n(X)$.

(c) Suppose that X is an H-space, i.e., that the fold map $\nabla\colon X \vee X \to X$ extends to a map $\mu\colon X \times X \to X$ in the pointed homotopy category hTop_*. Show that all Whitehead products on $\pi_{*+1}(X)$ are zero.

(d) Show that the suspension map $\pi_{p+q-1}(X) \to \pi_{p+q}(\Sigma X)$ sends all Whitehead products to zero.

Hint. Loop spaces are H-spaces.

Remark. One can check that $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}]$ equals $\pm 2\eta \in \pi_3(S^2)$, which gives another proof that $\Sigma\eta$ is 2-torsion.